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Graph Theory and Applications

J. AKIYAMA
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GRAPH THEORY AND APPLICATIONS

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Hakone, Japan, June 1–5, 1986

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Guest Editors: J. AKIYAMA, Y. EGAWA and H. ENOMOTO

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PREFACE

In the 1970s, there were few graph theorists among the 5000 members of the Japan Mathematics Society. Graph Theory was not even considered a suitable area in which to write a Ph.D. dissertation.

The situation has changed over the last few years; through the efforts of a small but determined group of mathematicians, Graph Theory has flourished in Japan and Japanese graph theorists have received worldwide acceptance. The phenomenal success of the First Japan Conference of Graph Theory and Applications held in Hakone, Japan, June 1–5, 1986 is an affirmation of this acceptance.

The conference was attended by 200 participants representing 22 countries. This volume contains the Proceedings of that conference.

The overwhelming response we have received from graph theorists all over the world encourages us to plan a Second Conference in Hakone, to precede the IMU Conference, scheduled to be held in Kyoto, Japan in 1990.

Jin Akiyama
Yoshimi Egawa
Hikoe Enomoto
for
the Organizing Committee

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HIGHLY IRREGULAR m-CHROMATIC GRAPHS

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A graph is highly irregular if it is connected and the neighbors of each vertex have distinct degrees. In this paper, we study existence and extremal problems for highly irregular graphs with a given maximum degree and focus our attention on highly irregular graphs that are m-chromatic for $m \geq 2$.

1. Introduction

One of the most widely studied classes of graphs are regular graphs. They show up in many contexts. For example, r-regular graphs with connectivity and edge-connectivity equal to r play a key role in designing reliable communication networks [2]. In the study of line graphs, Ray-Chaudhuri [6] showed that $L(G)$ is regular if and only if G is a regular graph or a semi-regular bipartite graph. Cages, which are minimum order r-regular graphs with girth g, have been widely studied—see the survey by Wong [7]. Godsil and McKay [4] examined graphs with the property that for each vertex v, the subgraph induced by the neighbors of v is regular and the subgraph induced by the nonneighbors of v is regular. They showed that such graphs are semi-regular (there are only two degrees) and the subgraph induced by the vertices of each degree are regular.

In studying regular graphs, neighbors and neighborhoods often play a vital role. Indeed, a connected r-regular graph can be defined as a connected graph for which the neighbors of each vertex all have the same degree r. We study graphs at the opposite extreme from regular graphs. In [1] a connected graph is defined to be *highly irregular* if for each vertex v, all the neighbors of v have distinct degrees. In Fig. 1, we display all the highly irregular graphs of order at most 8. Of the seven graphs in Fig. 1, all but one is bipartite. This very high proportion of bipartite graphs does not continue. For example, of the 16 highly irregular graphs

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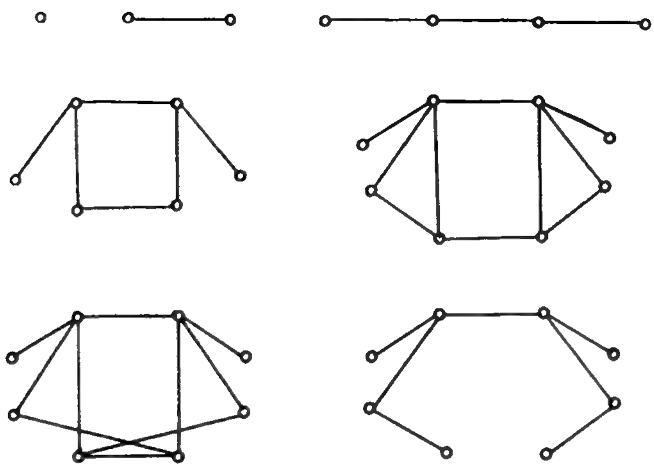

Fig. 1. The highly irregular graphs with at most eight vertices.

of order 9 or 10, eight are bipartite. When we refer to a *Δ-graph* (or Δ-tree) we mean a graph with maximum degree Δ. We shall first examine existence and extremal problems for cycles in highly irregular graphs, focusing on bipartite graphs.

2. Highly irregular trees

In this section we determine the values of n for which there is a highly irregular tree of order n. Several preliminary observations from [1] will be useful.

Remark 1. A vertex of degree Δ in a highly irregular Δ-graph is adjacent to exactly one vertex of each degree k, $1 \leq k \leq \Delta$.

It follows easily from Remark 1 that a highly irregular Δ-graph has order at least 2. A most useful property concerning the maximum degree Δ is the following.

Remark 2. Let G be a highly irregular Δ-graph on at least four vertices. Then G contains an induced path P_4 whose internal vertices have degree Δ in G and endpoints have degree 1 in G.

In what follows, we consider our (nontrivial) graphs to be doubly rooted at some pair of vertices u and v of degree Δ. We note that 4-trees are especially important in chemical applications [5]. Let T be a highly irregular Δ-tree of minimum order. Then the maximal subtree T_u containing u but not v clearly has the same order as the maximal subtree T_v containing v but not u (see Fig. 2). The

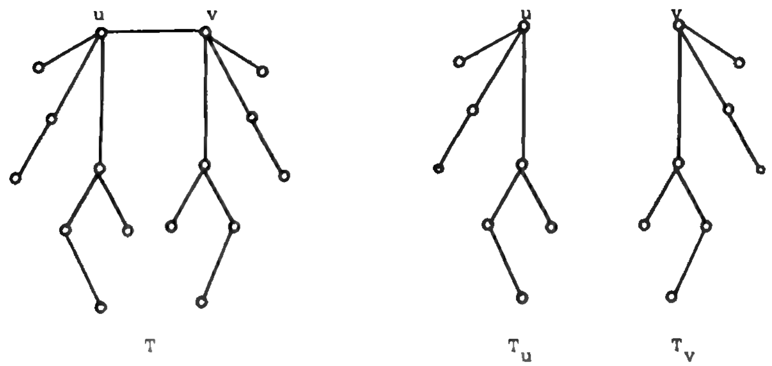

Fig. 2. Highly irregular 4-tree T with its subtrees T_u and T_v.

tree T has only two vertices of degree Δ. Otherwise, suppose T_u has a vertex of degree Δ, and let x be the nearest such vertex to u. Let y be the neighbor of x on the unique path from u to x. Then $\deg y \leq \Delta - 1$ and the maximal subtree containing x and not y can be replaced by a tree of smaller order so that $\deg x \leq \Delta - 1$. Then T is not minimum, a contradiction.

Theorem A. *The order of a highly irregular Δ-tree is at least* 2^{Δ}.

Highly irregular graphs were shown in [1] to exist for all orders except $n = 3$, 5, and 7. We shall give the corresponding results for trees. First, we need two lemmas. Since each vertex of degree Δ has exactly one neighbor of each degree k, $1 \leq k \leq \Delta$, maximum degree vertices occur in pairs.

Lemma 1. *In any nontrivial highly irregular Δ-graph, there is an even number of vertices of degree Δ.*

An easy observation is the following.

Lemma 2. *The highly irregular Δ-tree of order* 2^{Δ} *is unique.*

Theorem 1. *There exists a highly irregular tree for every order except* $n = 3$, 5, 6, 7, 11, 12, *and* 13.

Proof. The result for $n \leq 8$ follows from Fig. 1. To obtain a highly irregular tree of order 9, subdivide one of the edges joining a vertex of degree 2 with a vertex of degree 1 in the last graph of Fig. 1. For the tree of order 10, subdivide the other such edge as well.

Next we show that orders 11, 12, and 13 are not possible. By Theorem A, Δ is at most 3 for these values of n. In the tree of order 10, no additional subdivisions

are possible without losing the highly irregular property. Thus, when $11 \leqslant n \leqslant 15$, a highly irregular tree of order n must have at least three vertices of degree 3, and by Lemma 1 it must have at least four such vertices. Also, for T to be highly irregular, there must be a 'buffer' of two degree 2 vertices between a pair of nonadjacent degree 3 vertices. This forces at least the graph in Fig. 3. Thus there are no highly irregular trees of order 11, 12, or 13. Fig. 3 shows the tree of order 14, and a tree of order 15 is obtained by subdividing edge xy.

Let $n \geqslant 16$, let $\Delta = \lfloor \log_2 n \rfloor$, and let $x = n - 2^{\Delta}$. To construct a highly irregular tree T^* of order n, begin with the unique highly irregular Δ-tree T of order 2^{Δ} doubly rooted at u and v. Note that the number x of additional vertices needed to produce T^* is less than 2^{Δ}. Let u_k and v_k be the neighbors of u and v with degree k. Then let a and b be the vertices of degree 1 adjacent to u_2 and v_2, respectively, let c be the vertex of degree 1 at greatest distance from u in the branch at u of T_u containing u_3, and let e be the vertex of degree 1 adjacent to v_3 (see Fig. 4, Step 1). In Fig. 4,

$$n = 49, \qquad \text{so } \Delta = \lfloor \log_2 49 \rfloor = 5 \qquad x = 49 - 2^5 = 17 = 6(2) + 5$$

Let the number of additional vertices needed to produce T^* be $x = 6y + t$, where $y, t \in \{0, 1, 2, \ldots\}$ and $0 \leqslant t \leqslant 5$. If $y > 0$, attach a string of y minimum order highly irregular 3-trees at vertex a so that vertex a has degree 2 (see Fig. 4, Step 2). If $t \geqslant 3$, attach a path P_4 via a center vertex at d; if $t = 1$ or 4, attach a pendant edge at b; and if $t = 2$ or 5, attach pendant edges at both b and c (see Fig. 4, Step 3). The resulting tree is highly irregular and has n vertices. □

In certain problems, for example in chemical applications of graph theory, one wants to restrict the value of Δ. If we limit Δ to 3, we can construct a highly irregular tree for $n = 8$ and then for larger n by either subdividing at most two possible edges or adjoining an additional tree of order 6 containing two more degree 3 vertices (see Fig. 3). This process can be repeated and all highly irregular 3-trees can be obtained in this manner. Therefore, three consecutive orders for which highly irregular 3-trees exist are followed by three consecutive orders for which there is no highly irregular 3-tree, and visa versa. Thus, for infinitely many values of n, there is no highly irregular 3-tree of order n. When $\Delta \geqslant 4$, there are Δ-trees on n vertices for all $n \geqslant 2^{\Delta}$. In fact, such Δ-trees can be formed using exactly two vertices of degree Δ.

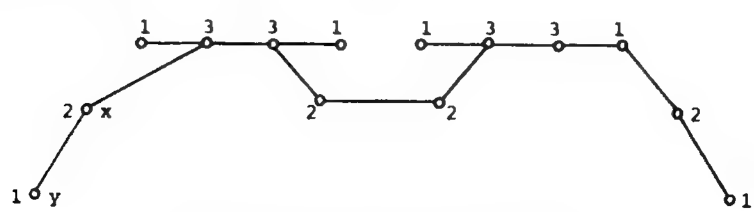

Fig. 3. The highly irregular tree of order 14.

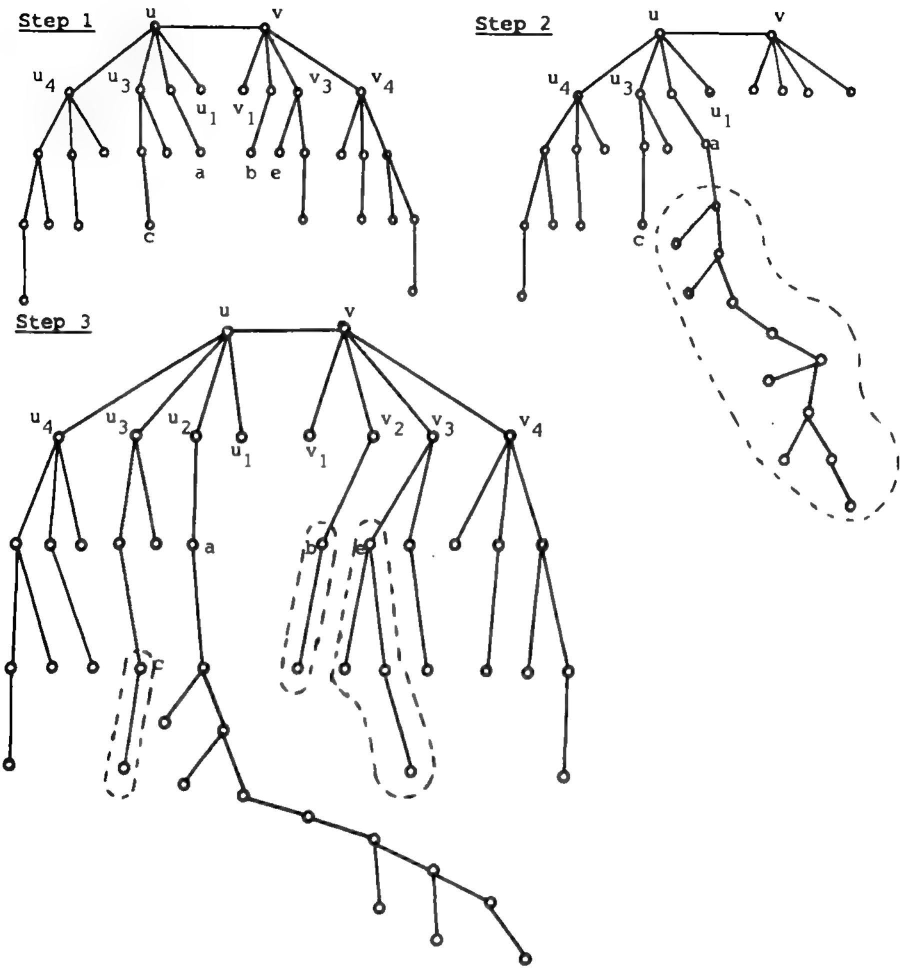

Fig. 4. Construction of a highly irregular tree of order 49.

Theorem 2. *There is a highly irregular* 3*-tree of order n if and only if* $n \equiv 2, 3$, *or* 4 (*mod* 6), *where* $n \geqslant 8$. *For* $\Delta \geqslant 4$, *a highly irregular* Δ*-tree of order n exists if and only if* $n \geqslant 2^{\Delta}$.

Note that the highly irregular 3-trees are all caterpillars. We denote the family of highly irregular Δ-graphs by $\mathcal{G}_\Delta$. It is easily observed that $\mathcal{G}_0 = \{K_1\}$, $\mathcal{G}_1 = \{K_2\}$, and $\mathcal{G}_2 = \{P_4\}$. We show in the next section that besides the 3-trees of Theorem 2, the family $\mathcal{G}_3$ of all highly irregular graphs with maximum degree 3 contains only one other type of graph: highly irregular unicyclic graphs with girth $4x$.

3. Bipartite graphs

We now turn our attention to bipartite graphs that are not necessarily trees. Recall that the *girth* of a graph is the length of its smallest cycle. When $\Delta = 3$ or 4, we would like to know the minimum order of a highly irregular bipartite graph with girth $g = 2t$. A *unicyclic* graph is a connected graph containing exactly one cycle. The next result shows that a unicyclic highly irregular graph with maximum degree 3 is necessarily bipartite.

Theorem 3. *A unicyclic highly irregular graph with $\Delta = 3$ and girth g exists if and only if $g \equiv 0$ (mod 4).*

Proof. Suppose G is a unicyclic highly irregular graph with $\Delta = 3$ and girth g, and let C be the cycle in G. Then each vertex v in C has degree 2 or 3, and one neighbor of v on C has degree 2 and the other has degree 3. Thus, the degree 3 vertices on C come in consecutive pairs, as do the degree 2 vertices, and the pairs alternate as we proceed around the cycle. This implies that there is the same number x of degree 3 pairs on C as degree 2 pairs. Thus $g = 4x$, so $g \equiv 0$ (mod 4).

If $g \equiv 0$ (mod 4), we construct a unicyclic highly irregular graph with girth g and $\Delta = 3$ as follows: Let $C = v_1, v_2, \ldots, v_g, v_1$ and attach a pendant edge at each vertex v_{4k+1} and v_{4k+2}. □

Since the neighbors of each vertex of degree 3 in Theorem 3 have degree 3 and degree 2 on C, its neighbor of degree 1 is not on C. The distribution of degree 3 vertices on C then makes G uniquely determined.

Corollary 1. *For each positive integer $g \equiv 0$ (mod 4), there is precisely one unicyclic highly irregular graph with $\Delta = 3$ and girth g, and it has order $\frac{1}{2}(3g)$.*

Corollary 2. *A highly irregular graph with $\Delta = 3$ that is not a tree is unicyclic.*

If G is a bipartite graph with k vertices in each part, then we call G a $k \times k$ *bipartite graph.*

Corollary 3. *For all $k \geqslant 3$, there is a highly irregular $k \times k$ bipartite graph with $\Delta = 3$.*

Proof. The unicyclic graphs with girth $g \equiv 0$ (mod 4) in Corollary 1 have order $n = 3g/2$. Since $g \equiv 0$ (mod 4), it follows that $n = 3g/2 \equiv 0$ (mod 6). In Theorem 2, we saw that there are highly irregular 3-trees with order $n \equiv 2$ or 4 (mod 6). In all cases, the resulting graph are $k \times k$ bipartite. □

Corollary 4. *Let G be a highly irregular $k \times k$ bipartite graph with $\Delta = 3$. If $k \equiv 1$ or 2 (mod 3), then G is a unique tree. If $k \equiv 0$ (mod 3), then G is unique and unicyclic with a cycle of length $4k/3$.*

The following remark is easy to prove.

Remark 3. The minimum order of a highly irregular bipartite Δ-graph is 2 and the graph achieving the minimum is the unique highly irregular $\Delta \times \Delta$ bipartite Δ-graph.

When $\Delta \geq 3$, the graphs in Remark 3 all have girth 4. Thus, the minimum order of highly irregular bipartite Δ-graphs with girth 4 is 2Δ.

We now restrict our attention to highly irregular bipartite 4-graphs. By Remark 3, the minimum order of such a graph is 8. If, additionally, we require that G be a tree, Theorem A and Lemma 2 imply that the minimum order is 16. The graph of order 8 has girth 4 and the tree has infinite girth. Thus, we consider the intervening girths. Since G is bipartite, each cycle (and therefore the girth) of G is even. Let G be a bipartite graph with $\Delta = 4$ and girth $g = 2t$, and let C be a cycle of length g in G. It is easy (but tedious) to show that if there is a vertex of degree 4 not on C, then the order of G is at least $\lfloor \frac{1}{2}(3g) \rfloor + 6$ when $g \geq 6$.

Theorem 4. *Let G be a highly irregular bipartite 4-graph with girth $g < \infty$. Then the minimum order of G is*

$$\begin{cases} 8 & \text{if } g = 4, \text{ and} \\ g + 2\lfloor \frac{1}{4}g \rfloor + 6 & \text{if } g \geq 6. \end{cases}$$

Proof. Fig. 1 shows that the statement is true when $g = 4$, and the case for $g = 6$ can be easily checked. Thus, suppose $g \geq 8$. We first show that a minimum order graph is unicyclic. Suppose this is not the case, and let C_2 be a cycle in addition to a first cycle C_1 with g vertices. Then C_2 either shares some vertices with C_1 or is joined by some path to C_1 (Fig. 5). In either case, C_2 introduces at least $\frac{1}{2}g - 1$ additional vertices. Therefore, from C_1 and C_2 there are at least $\frac{1}{2}(3g) - 1$ vertices and at most half of these have degree 2. Thus at least $\lceil \frac{1}{4}(3g - 2) \rceil$ vertices on $C_1 \cup C_2$ have degree greater than 2. Since C_1 and C_2 meet in at most two vertices, at least $\lceil \frac{1}{4}(3g - 2) \rceil - 2$ vertices have neighbors not on $C_1 \cup C_2$. G has at least two vertices of degree 4. This produces at least six vertices not on C_1 or C_2 (two of which have been counted already, thus an actual increase of 4). Thus the order of G is at least

$$\tfrac{1}{2}(3g) - 1 + \lceil \tfrac{1}{4}(3g - 2) \rceil - 2 + 4.$$

Since $g \geq 8$, $\lceil \frac{1}{4}(3g - 2) \rceil \geq 6$, so the order of G is at least $\frac{1}{2}(3g) + 7$ if there are two or more cycles. However, we can construct a unicyclic, highly irregular bipartite

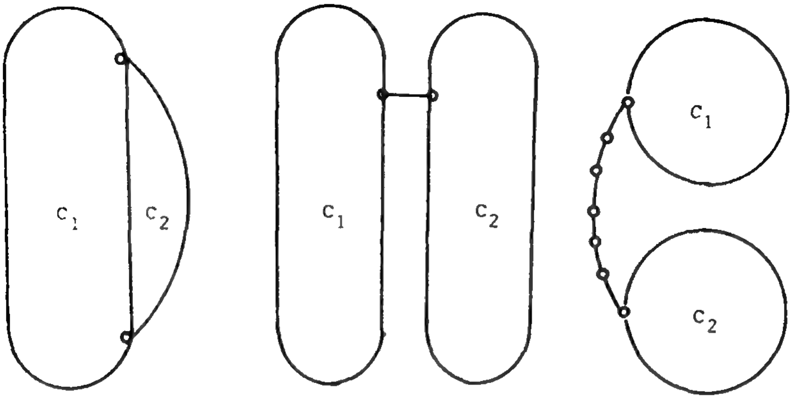

Fig. 5. Some possibilities for C_1 and C_2.

4-graph with girth g and order $g+2\lfloor\frac{1}{4}g\rfloor+6\leq\frac{1}{2}(3g)+7$. Thus, the minimum order graphs are unicyclic.

To construct a minimum graph when $g\geq 6$, begin with a cycle $C: v_1, v_2, \ldots, v_g, v_1$. Let the degrees in G be $\deg v_1=\deg v_g=4$. If $g\equiv 0 \pmod 4$, then $\deg v_k=3$ for $k\equiv 2$ or $3 \pmod 4$ and $\deg v_k=2$ for $k\equiv 0$ or $1 \pmod 4$, $1<k<g$. If $g\equiv 2 \pmod 4$, then $\deg v_2=\deg v_{g-1}=3$, $\deg v_k=2$ for $k\equiv 0$ or $3 \pmod 4$, and $\deg v_k=3$ for $k\equiv 1$ or $2 \pmod 4$, $3\leq k\leq g-2$ (see Fig. 6). Finally, in either case, attach a pendant edge at each vertex of degree 3 and attach a P_4 via one of its center vertices at v_1 and v_g. The resulting highly irregular bipartite 4-graph has girth g and order $g+2\lfloor\frac{1}{4}g\rfloor+6$. □

We have characterized the graphs in the families $\mathscr{G}_0$, $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$, and in Theorem 4 we determined the minimum order for bipartite graphs in $\mathscr{G}_4$. It seems to be a difficult task to characterize graphs in $\mathscr{G}_\Delta$ for $\Delta\geq 4$. We can, however, determine the orders for which there is a graph in $\mathscr{G}_\Delta$, $\Delta\geq 4$.

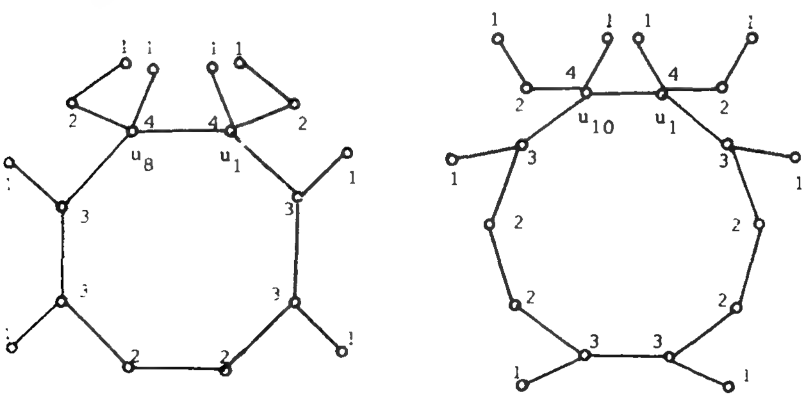

Fig. 6.

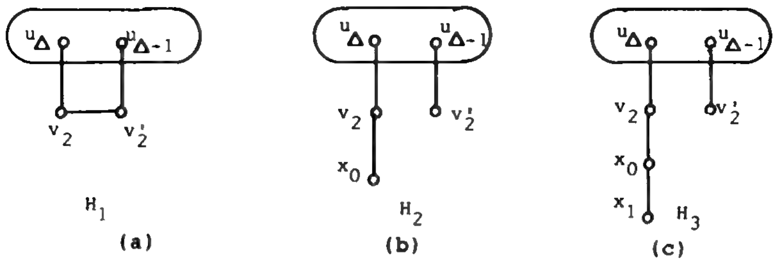

Fig. 7.

Theorem 5. *For any pair of integers n and Δ such that $n \geq 2\Delta \geq 8$, there exists a graph of order n in $\mathcal{G}_\Delta$.*

Proof. Suppose $n = 2\Delta \geq 8$. Let H_0 denote the unique highly irregular bipartite Δ-graph with order 2Δ described in Remark 3. Let $V = \{v_1, v_2, \ldots, v_\Delta\}$ and $U = \{u_1, u_2, \ldots, u_\Delta\}$ be its parts, where $\deg u_k = \deg u_k = k$. Next, we describe a sequence $\langle H_t \rangle$ of graphs in $\mathcal{G}_\Delta$ such that the order of H_t is $2\Delta + t$, $t = 1, 2, \ldots$. The graph H_1 is obtained from H_0 by subdividing the edge $v_2u_{\Delta-1}$ with vertex v_2' (see Fig. 7a). The graph H_2 is formed from $H_1 - v_2v_2'$ by adding vertex x_0 together with the edge v_2x_0 (see Fig. 7b). The graph H_3 is formed from H_2 by adding vertex x_1 together with edge x_0x_1 (see Fig. 7c).

The graph H_4 is obtained from H_2 by deleting u_2 and adding the vertices w_0, z_0 and z_1, together with the edges $v_{\Delta-1}w_0$, $v_\Delta z_0$, and z_0z_1. In a similar manner, we

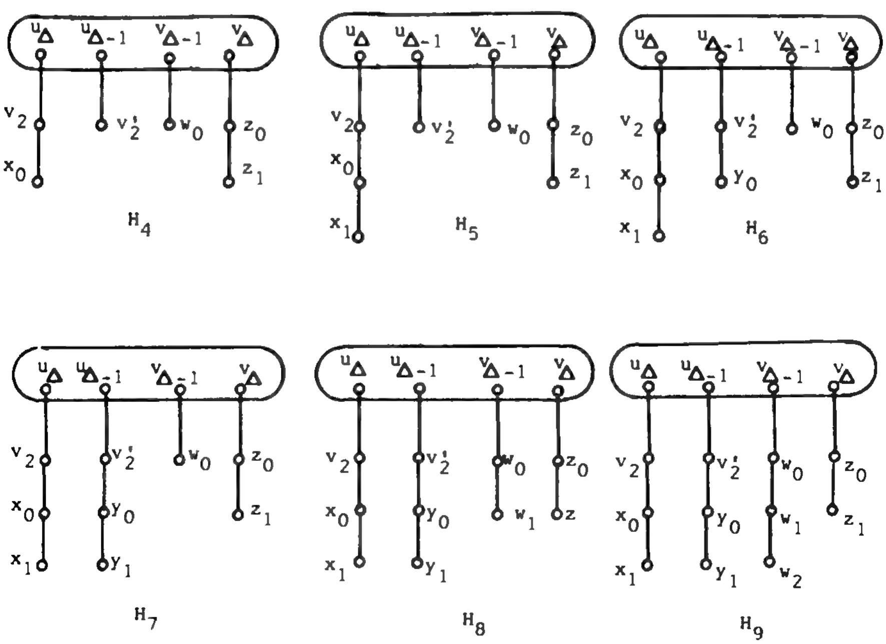

Fig. 8.

construct the graphs H_5, H_6, H_7, H_8, H_9, which along with the graph H_4 are illustrated in Fig. 8.

Since the set $B = \{4, 5, 6, 7, 8, 9\}$ forms a complete residue system modulo 6, every integer $n \geq 10$ can be expressed in a unique way as $n = 2\Delta + j + 6k$, for some $j \in B$ and $k \geq 1$. Thus the graph H_n is obtained by attaching a string of k minimum order highly irregular 3-trees at vertex z_1 in H_j. Each such H_n is highly irregular and has maximum degree Δ. $\square$

It is easy to show that there is no bipartite graph in $\mathscr{G}_\Delta$ of order $2\Delta + 1$, which gives the following.

Corollary 5. *For a given positive integer $\Delta \geq 4$, there exists a bipartite graph of order n in $\mathscr{G}_\Delta$ if and only if $n = 2\Delta$ or $n \geq 2\Delta + 2$.*

4. *m*-Chromatic graphs

Corollary 5 has the following generalization kindly communicated to us by Chartrand.

Theorem 6. *Let G be a highly irregular m-chromatic Δ-graph, $m \geq 3$. Then $\Delta \geq 2m - 2$.*

Proof. As noted earlier, all graphs in $\mathscr{G}_k$, $0 \leq k \leq 3$, are bipartite, so $\Delta \geq 4$. If $m = 3$, then $\Delta \geq 4 = 2m - 2$.

Suppose that $m \geq 4$. Let H be a critically m-chromatic subgraph of G. If $H \cong K_m$, then each vertex of H is adjacent to $m - 1$ vertices of H (having distinct degrees in G), so that H contains a vertex u with $\deg_G u \geq 2m - 2$, which implies that $\Delta \geq 2m - 2$.

Finally, suppose that $H \not\cong K_m$. Since $m \geq 4$, H is not an odd cycle (which is only 3-chromatic). By a result of Brooks [3], H contains a vertex w of degree at least m. Since $\delta(H) \geq m - 1$, w is adjacent to a vertex of degree (in G) at least $2m - 2$ so that $\Delta \geq 2m - 2$. $\square$

Theorem 7. *For given integers m (≥ 3) and Δ, with $\Delta \geq 2m - 2$, there exists an integer $f(m, \Delta)$ such that whenever $n \geq f(m, \Delta)$ there is an m-chromatic graph G in $\mathscr{G}_\Delta$ having order n.*

Outline of proof. Begin with the graph H_0 with vertex set

$$V(H_0) = \{a_1, a_2, \ldots, a_{m-1}, b_1, b_2, \ldots, b_{m-1}, c_1, c_2, \ldots, c_{m-1}, e_1, e_2, \ldots, e_{m-1}\}$$

and edge set

$$E(H_0) = \{b_j b_k, c_j c_k, b_j c_j : j \neq k\} \bigcup \{b_j a_k, c_j e_k : j \geq k\}.$$

This graph is a highly irregular, m-chromatic graph of order $4m-4$ with $\Delta = 2m-2$. For this fixed value of Δ, the graphs for values of $n > 4m-4$ are obtained using operations analogous to those used in the proof of Theorem 5. By using similar techniques with slightly more complicated constructions, we can obtain the graphs with $\Delta > 2m-2$. □

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EXPLICIT CONSTRUCTION OF LINEAR SIZED TOLERANT NETWORKS

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For every $\epsilon > 0$ and every integer $m > 0$, we construct explicitly graphs with $O(m/\epsilon)$ vertices and maximum degree $O(1/\epsilon^2)$, such that after removing any $(1-\epsilon)$ portion of their vertices or edges, the remaining graph still contains a path of length m. This settles a problem of Rosenberg, which was motivated by the study of fault torerant linear arrays.

1. Introduction

What is the minimum possible number of vertices and edges of a graph G, such that even after removing all but ϵ portion of its vertices or edges, the remaining graph still contains a path of length m? This problem arises naturally in the study of fault tolerant linear arrays, (see [18]). The vertices of G represent processing elements and its edges correspond to communication links between these processors. If p, $0<p<1$ is the failure rate of the processors, it is desirable that after deleting any p portion of the vertices of G, the remaining part still contains a (simple) path (= linear array) of length m. Similarly, if μ, $0<\mu<1$ denotes the failure rate of the communication links, it is required that after deleting any μ portion of the edges of G the remaining part still contains a relatively long path. The objective is to construct such graphs G with a small number of vertices and edges, since these will give rise to efficient networks. Some variants of this problem are discussed in [14, 9, 19]. In this note we prove the following result.

Theorem 1.1. *For every $\epsilon > 0$ and every integer $m \geq 1$ there is a graph G, which can be explicitly constructed, with $O(m/\epsilon)$ vertices and maximum degree $O(1/\epsilon^2)$, such that even after deleting all but ϵ-portion of its vertices or all but ϵ-portion of its edges, the remaining graph still contains a path of length m.*

This settles the problem raised by Rosenberg [18], and is also related to the

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study of size Ramsey numbers (see [10, 4]). We note that the edge version of the above theorem for $\epsilon = \frac{1}{2}$ was proved by Beck [4], and his proof can be modified to give the general case, without explicit construction. Our main contribution here is to obtain an explicit construction by combining the arguments of Beck [4] with some of the eigenvalues technique of [1–3] and the recent construction of expanders given by Lubotzky et al. [16].

2. The proof of the main result

For a graph $H = (V, E)$ and $S \subseteq V$, let $N_H(S)$ denote the set of all neighbors in H of vertices of S. Pósa [17] proved the following useful lemma which provides a relation between the expanding properties of a graph and the size of the longest path it contains. (See also [15] for a simpler proof.)

Lemma 2.1. *Let $H = (V, E)$ be a nonempty graph. If $|N_H(S) - S| \geqslant 2|S| - 1$ for every vertex subset $S \subseteq V$ of cardinality $|S| \leqslant k$, then H contains a path of length $3k - 2$.*

The following simple lemma is an old folklore result (see, e.g. [4]):

Lemma 2.2. *Any graph G on n vertices with averge degree d contains an induced subgraph H such that for every vertex-set S of H the number of edges incident to vertices in S is at least $d|S|/2$.*

Next we need a relation between the eigenvalues of a graph G and the density of its induced subgraphs. Let $A = A_G$ be the n by n adjacency matrix of a d-regular graph $G = (V, E)$ on $V = \{1, 2, \ldots, n\}$. Clearly d is the largest eigenvalue of A and its eigenvector is the all 1 vector. Suppose that the absolute value of any other eigenvalue of A is at most λ. For $S \subseteq V$ let $e(S, \bar{S})$ denote the number of edges of G between vertices of S and vertices of $\bar{S} = V - S$, and let $e(S)$ denote the number of edges that join two vertices of S.

Lemma 2.3. *In the above notation, for every subset $S \subseteq V$ of cardinality $|S| = \alpha n$*

$$|e(S) - \tfrac{1}{2}d\alpha^2 n| \leqslant \tfrac{1}{2}\lambda\alpha(1 - \alpha) \cdot n. \tag{2.1}$$

We note that the term $\frac{1}{2}d\alpha^2 n$ is roughly the expected number of edges in an induced subgraph of G of size $\alpha \cdot n$. Thus, for small λ, every such induced subgraph has about the same number of edges.

Proof. Define a vector $f: V \to R$ by $f(i) = -1/|S|$ if $i \in S$ and $f(i) = 1/(n - |S|)$ if $i \notin S$. Since $\sum_{i=1}^{n} f(i) = 0$, i.e. f if orthogonal to the eigenvector of the largest

eigenvalue of A, we conclude that $|(Af, f)| \leq \lambda(f, f)$, where $(,)$ is the usual scalar product. One can easily check that

$$(Af, f) = 2 \sum_{ij \in E} f(i) \cdot f(j) = d \sum_{i=1}^{n} f^2(i) - \sum_{ij \in E} (f(i) - f(j))^2.$$

For the specific f defined above

$$\sum_{i=1}^{n} f^2(i) = \frac{1}{|S|} + \frac{1}{n - |S|} \quad \text{and} \quad \sum_{ij \in E} (f(i) - f(j))^2 = e(S, \bar{S})\left(\frac{1}{|S|} + \frac{1}{n - |S|}\right)^2.$$

Thus

$$\left| e(S, \bar{S})\left(\frac{1}{|S|} + \frac{1}{n - |S|}\right)^2 - d\left(\frac{1}{|S|} + \frac{1}{n - |S|}\right) \right| \leq \lambda\left(\frac{1}{|S|} + \frac{1}{n - |S|}\right),$$

which implies, by substituting $|S| = \alpha n$, that

$$|e(S, \bar{S}) - d\alpha(1 - \alpha)n| \leq \lambda\alpha(1 - \alpha) \cdot n. \tag{2.2}$$

Since G is a d-regular graph,

$$2e(S) + e(S, \bar{S}) = d\,|S| = d\alpha n,$$

i.e.

$$e(S) = \tfrac{1}{2}d\alpha n - \tfrac{1}{2}e(S, \bar{S}).$$

This and inequality (2.2) imply inequality (2.1). This completes the proof. □

Proof of Theorem 1.1. Lubotzky et al. [16] showed that if p and q are primes congruent to $1 \bmod 4$, with p a quadratic non-residue $\bmod q$, then there is an explicitly constructed $d = p + 1$ regular graph G with $n = q(q^2 - 1)/2$ vertices, such that the absolute value of each of its eigenvalues but the first is at most $\lambda = 2\sqrt{d - 1}$. We next show that for properly chosen p and q, G satisfies the assertion of Theorem 1.1. We first consider the case of deleting vertices. Suppose we delete all but a set V of $\epsilon \cdot n$ vertices of G. By Lemma 2.3 the induced subgraph of G on V contains at least $\frac{1}{2}d\epsilon^2 n - \frac{1}{2}\lambda\epsilon(1 - \epsilon)n$ edges, i.e. has average degree at least $\epsilon d - \lambda(1 - \epsilon)$. By Lemma 2.2 this graph contains an induced subgraph H in which every vertex set of cardinality x hits at least $\frac{1}{2}(\epsilon d - \lambda(1 - \epsilon)) \cdot x$ edges. Let S be an arbitrary vertex subset of H, of cardinality $x = \beta n \leq \alpha n$, where $\alpha < \frac{1}{9}\epsilon$ will be chosen later. We next show that for a properly chosen d:

$$|N_H(S) - S| > 2\,|S|. \tag{2.3}$$

Indeed, otherwise, if $T = N_H(S) - S$, then $|S \cup T| \leq 3x$ and there are in H (and hence in G) at least $\frac{1}{2}(\epsilon d - \lambda(1 - \epsilon)\beta) \cdot n$ edges joining vertices of $S \cup T$. However, by Lemma 2.3

$$e(S \cup T) \leq \tfrac{1}{2}d9\beta^2 n + \tfrac{1}{2}\lambda \cdot 3\beta(1 - 3\beta) \cdot n$$

and therefore the inequality

$$\epsilon d - \lambda(1 - \epsilon) \leq 9\beta d + 3\lambda(1 - 3\beta)$$

must hold. Since $\lambda \leqslant 2\sqrt{d-1}$ this implies

$$d \leqslant \frac{2\sqrt{d-1}(4-\epsilon-9\beta)}{(\epsilon-9\beta)} \tag{2.4}$$

Hence if we choose d such that

$$d > 4 \cdot \left(\frac{4-\epsilon-9\alpha}{\epsilon-9\alpha}\right)^2 \tag{2.5}$$

then (2.4) is violated for all $\beta \leqslant \alpha$ and hence (2.3) holds. By Lemma 2.1 we conclude that H contains a path of length $3\lfloor \alpha n \rfloor - 2$. Therefore if we choose e.g. $\alpha = \frac{1}{18}\epsilon$ and we choose the primes p and q in the construction of G such that

$$d = p + 1 > 4\left(\frac{8}{\epsilon}\right)^2 \quad \text{and} \quad n = \frac{q(q^2-1)}{2} \geqslant \frac{6}{\epsilon}(m+5) \tag{2.6}$$

we conclude that even if we delete all but $\epsilon \cdot n$ vertices of G, the remaining part still contains a path of length m. This completes the proof for the case of deleting vertices. The case of deleting edges is somewhat simpler. Indeed, if we delete all but an ϵ-portion of the edges we are left with a graph of average degree $\epsilon \cdot d$. This graph contains, by Lemma 2.2, an induced subgraph H in which any set of x vertices hits at least $\epsilon d \frac{1}{2} x$ edges. Hence if S is any vertex subset of cardinality $\beta n \leqslant \alpha n$ of H, where $\alpha < \frac{1}{9}\epsilon$, and if d satisfies $d > 4\left(\frac{3-9\alpha}{\epsilon - 9\alpha}\right)^2$ one can check, as before, that $|N_H(S) - S| > 2\,|S|$. Thus, by Lemma 2.1, H contains a path of length $3\lfloor \alpha n \rfloor - 2$. It is easy to check that the previous choice of p, q given in (2.6) suffices to guarantee a path of length m in this case, as well. (In fact, here a slightly smaller d is enough.) By the standard results about the distribution of primes (see e.g. [8]), there is a choice for p and q for which (2.6), as well as the estimates $n = O(m/\epsilon)$ and $d = O(1/\epsilon^2)$ hold. This completes the proof. □

3. Related problems

3.1. *Size Ramsey numbers*

The size Ramsey number $\lambda_s(G)$ of a graph G is the least number of edges in a graph H with the property that any two coloring of the edges of H contains a monochromatic copy of G. Size Ramsey numbers were first considered in [10], and several results on them can be found also in [6, 12, 13]. Beck's result [4], mentioned in Section 1, resolves the problem raised by Erdös of estimating the size Ramsey numbers for paths. Beck's construction, however, is probabilistic. Typically, explicit constructions are much more difficult to find than random ones for Ramsey type problems (see [13]). Our construction supplies an explicit example showing that the size Ramsey number for paths is linear.

3.2. *Fault tolerant graphs for bounded degree trees*

A natural extension of Theorem 1.1 is obtained by replacing the requirement for paths of length m by a requirement for all trees of maximum degree k and size m. Beck [4] proved, without an explicit construction, that there exists a graph G with $O(k \cdot m \cdot (\log m)^{12})$ edges, such that any set of half of its edges contains every tree of size m and maximum degree k. Very recently, Friedman and Pippenger [11] gave, for every $\epsilon > 0$ and $m, k \geqslant 2$, an explicit construction of a graph G with $O(mk^2/\epsilon)$ vertices and maximum degree $O(k^2/\epsilon^2)$ such that any set of an ϵ-portion of its edges contains every tree of size m and maximum degree k. Their construction is based on an interesting generalization of Pósa's theorem (Lemma 2.1) from paths to trees. Some other related results can be found in [5, 7].

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SORTING IN ROUNDS

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The main result of the paper is that if n is sufficiently large and $2 \leqslant r = r(n) \leqslant \log\log\log n$ then n objects can be sorted in r rounds by asking no more than $2^{r+1} n^{1+1/r} (\log n)^{2-2/r}/(\log\log n)^{(r-1)/r}$ questions in each round.

Suppose that we are given n elements in some total order unknown to us. We wish to identify this total order by asking questions in r rounds: we aks m_1 questions in the first round, then process the information obtained and in the second round we choose an appropriate set of m_2 questions, and so on, in the rth round was ask m_r questions such that the answers to our questions tell us the order of the elements. A question is just a binary comparison: how are a and b related? The answer tells us that either $a < b$ or $b < a$. Let us write SORT(n, r) for the minimal value of m for which we can always distribute our questions in such a way that $m_i \leqslant m$ for $i = 1, 2, \ldots, r$. Thus SORT(n, r) is the minimal number of parallel processors that can sort n elements in time r.

In studying SORT(n, r) the nature of the difficulties depends on the relationship between n and r: *whether* r is large, say at least $\log n$, or small, say less than $\log r$, perhaps even bounded. Ajtai, et al. [2, 3] proved that SORT$(n, \lfloor \log n \rfloor) \leqslant cn$ for an absolute constant c. Therefore, if $r \geqslant \log n$ then SORT$(n, r) \leqslant 2cn(\log n)/r$. Here we shall be concerned with the case in which r is small. This case has been studied by Häggkvist and Hell [12–14], Bollobás and Rosenfeld [10], Bollobás and Thomason [11], Alon [4], Alon, Azar and Vishkin [5] and Pippenger [18]. (For a review of these results, see [9] and [6, Ch. XV]. Related results can be found in [1, 7 and 8].) In particular, Häggkvist and Hell [13] proved that SORT$(n, r) = \Omega(n^{1+1/r})$ for every fixed r. This was strengthened by Alon et al. [5] who showed that SORT$(n, r) = \Omega(rn^{1+1/r})$ for $r = r(n) \leqslant \log n$ and SORT$(n, r) = \Omega(n^{1+1/r}(\log n)^{1/r})$ for every fixed r.

Concerning upper bounds, Bollobás and Thomason [11] proved SORT$(n, 2) \leqslant \frac{1}{2} n^{\frac{3}{2}} \log n$ if n is sufficiently large. Alon et al. [5] improved this to SORT$(n, 2) = 0(n^{\frac{3}{2}} \log n/(\log\log n)^{\frac{1}{2}})$. More importantly, Pippenger [18] proved that SORT$(n, r) = 0(n^{1+1/r}(\log n)^{2-2/r})$ for every fixed r. Even more, Pippenger used concrete sorting networks to sort the elements: he showed that the so-called *Ramanujan graphs* constructed recently by Lubotzky, Phillips and Sarnak [16–18] can be used as efficient sorting networks.

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In this paper we shall give an upper bound for $\mathrm{SORT}(n, r)$ in the case when r is constant or rather small compared to n. Similarly to the proof of most of the upper bounds, we shall use *random graphs*. (For an introduction to random graphs, see [6].) A weaker form of the main result of this paper was announced at the Conference on Combinatorics, Graph theory and Computing in Boca Raton in 1982. (The result was slightly misquoted in [9].)

Let V be the set of n elements we wish to sort. The questions asked in the first round form a graph G with vertex set V. The answers we get give us an acyclic orientation $\vec{G}$ of G and they tell us that the total order we are searching for is an extension of the partial order on V determined by $\vec{G}$. Our aim is to show that there is a graph G whose size $e(G)$ is rather small, which is such that in every acyclic orientation $\vec{G}$ of G the vertex set can be covered by sets $V_1, V_2, \ldots, V_s$ of size at most t, where both s and t are small, such that any two elements incomparable in the partial order determined by $\vec{G}$ belong to some V_i. Hence if in the remaining $r-1$ rounds we sort each V_i then V itself will be sorted. Therefore $\mathrm{SORT}(n, r) \leq \max\{e(G), s\,\mathrm{SORT}(t, r-1)\}$. We shall show that an appropriate random graph will do for G. First we introduce some graphs that will turn out to be suitable for sorting.

Let $a_0 = 1 < a_1 < \cdots < a_s < a$ and $1 < d_s < d_{s-1} < \cdots < d_1$. Let G be a graph with vertex set $V = V(G)$. We call G an *$(a; a_1, d_1; a_2, d_2; \ldots; a_s, d_s)$-graph* if

(i) for all $V_i \subset V$, $|V_i| \geq a$, there exists a set $W_i \subset U_i$, $|W_i| \geq |V \setminus V_i| - a$ such that if $X_i \subset V_i$ and $|X_i| \leq a_i$ then $|\Gamma(X_i) \cap V_i| \geq d_i\,|X_i|$

(ii) for all $A_s \subset V$, $|A_s| \geq a_s$, we have $|V \setminus (\Gamma(A_s) \cup A_s)| \leq a$.

Note that both (i) and (ii) express the fact that G is 'spreading'. Condition (i) says that a given large subset V_i of vertices, all not-too-large subsets in a large subset of $V \setminus V_i$ have many neighbours in the second. Condition (ii) is even simpler: the complement of G does not contain a $K(a, a_s)$, a complete bipartite graph with a vertices in one class and a_s in the second.

Let us see what happens if we use an $(a; a_1, d_1; \ldots; a_s, d_s)$-graph to sort n elements.

Lemma 1. *Let G be an $(a; a_1, d_1; \ldots; a_s, d_s)$-graph, let $\vec{G}$ be an acyclic orientation of G and set*

$$u = \left\lceil \sum_{i=1}^{s} \log(a_i/a_{i-1})/\log d_i \right\rceil + 1,$$

where $a_0 = 1$. Then in $\vec{G}$ all but at most $h + 2ua$ vertices dominate at least h vertices.

Proof. Take a complete order on $V = V(G)$ consistent with $\vec{G}$. Let T be the set of the top $n - (h + (2u-1)a)$ vertices, A_1 the next $2a$ vertices, A_2 the next $2a$ vertices, etc., up to A_{u-1}, and then let B be the bottom $h + a$ vertices of V. Thus

$T\cup A_1\cup A_2\cup\cdots\cup A_{u-1}\cup B$ is a partition of V. For simplicity, we also write $A_0=T$ and $A_u=B$.

Let u_1 be the minimal integer satisfying $d_1^{u_1}\geq a_1$, let u_2 be the minimal integer satisfying $d_1^{u_1}d_2^{u_2}\geq a_2$, etc., up to u_s. Then $\sum_{i=1}^s u_i\leq u-1$. For $1\leq j\leq u-1$ let $i(j)$ be the integer defined by

$$\sum_{i=1}^{i(j)-1} u_i<j\leq\sum_{i=1}^{i(j)} u_i.$$

Thus in the sequence $d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2, d_3, d_3, \ldots,$ with d_i repeated u_i times, the jth term is $d_{i(j)}$.

Define sets $\tilde{A}_j\subset A_j$, $|\tilde{A}_j|\geq|A_j|-a$, $j=u-1, u-2, \ldots, 0$, as follows. Set $\tilde{A}_{u-1}=A_{u-1}$. Having defined $\tilde{A}_{u-1}, \tilde{A}_{u-2}, \ldots, \tilde{A}_j$, $j\geq 1$, set $i=i(j)$ and $V_i=\tilde{A}_j$. Let $W_i\subset V\setminus V_i$, $|W_i|\geq|V\setminus V_i|-a\geq a$, be the set whose existence is guaranteed by condition (i), and set $\tilde{A}_{j-1}=W_i\cap U_i$.

By construction, $|\tilde{A}_0|\geq|T|-a=n-2ua-h$, so to prove the lemma it suffices to show that every vertex in $\tilde{A}_0$ dominates at least h vertices in the partial order given by $\vec{G}$. Let $x\in\tilde{A}_0$ and let X_i be the set of vertices in $\tilde{A}_i$ dominated by x. Then, by the definition of the sets $\tilde{A}_i$ we have

$$\begin{aligned}&|X_1|\geq d_1, |X_2|\geq d_2, \ldots, |X_{u_1}|\geq d_1^{u_1}\geq a_1, \ldots, |X_{u_1+u_2}|\\&\quad\geq d_1^{u_1}d_2^{u_2}\geq a_2, \ldots, |X_{u-1}|\geq d_1^{u_1}d_2^{u_2}, \ldots, d_s^{u_s}\geq a_s.\end{aligned}$$

Finally, by property (ii), X_{u-1} dominates all but at most a vertices in $B=Au$; in fact, all but at most a vertices in B are joined to at least d vertices in X_{u-1}. Hence x itself dominates all but at most a vertices in B so it dominates at least $|B|-a=h$ vertices. □

Before applying Lemma 1 to prove the inequality mentioned in the introduction, connecting sorting in r rounds to sorting in $r-1$ rounds, let us introduce some more terminology. Given an acyclic orientation $\vec{G}$ of a graph G with vertex set V, for $x\in V$ let $o_*(x)$ be the *minimal order of* x in an extension of $\vec{G}$ to a complete order and let $o^*(x)$ be the *maximal order of* x. Note that x dominates precisely $o_*(x)-1$ elements in $\vec{G}$ and is dominated by $n-o^*(x)$ elements, where $n=|V|$. In particular, $o_*(x)=1$ iff x is a minimal element of the partial order determined by $\vec{G}$ and $o^*(x)=n$ iff x is a maximal element. Lemma 1 claims that $|\{x\in V: o_*(x)\leq h\}|\leq h+2ua$ and, by symmetry, $|\{x\in V: o^*(x)>h\}|\leq n-h+2ua$.

Lemma 2. *Suppose there is an* $(a; a_1, d_1, \ldots, a_s, d_s)$*-graph* G *of order* n. *Set*

$$u=\left\lceil\sum_{i=1}^{s}\log(a_i/a_{i-1})/\log d_i\right\rceil+1$$

and let $v\geq 2\,ua$, *where* $a_0=1$. *Then*

$$\mathrm{SORT}(n, r)\leq\max\{e(G), n/2v\ \mathrm{SORT}(\lceil 4v\rceil, r-1)\}.$$

Proof. Let us use G to sort in the first round; say $\vec{G}$ is the acyclically oriented graph given by the answers to our questions. Set $j=\lfloor n/2v\rfloor$, $h_0=0$, $h_j=n$, and for $1\leqslant i\leqslant j-1$ define $h_i=(2i+1)v$. Note that $h_{j-1}+v>n$. Furthermore, for $1\leqslant i\leqslant j$ define

$$V_i=\{x\in V: o_*(x)\leqslant h_i \text{ and } o^*(x)>h_{i-1}\}.$$

We claim that

$$|V_i|\leqslant 4v \tag{1}$$

for all i, $1\leqslant i\leqslant j$. To see (1), define

$$B_i=\{x\in V: o_*(x)\leqslant h_i\}$$

and

$$T_i=\{x\in V: o^*(x)>h_{i-1}\}.$$

Then, by Lemma 1,

$$|B_i|\leqslant h_i+v \quad\text{and}\quad |T_i|\leqslant n-h_{i-1}+v.$$

Since $B_i\cup T_i=V$ and $V_i=B_i\cap T_i$, for $2\leqslant i\leqslant j-1$ we have $|V_i|\leqslant 2v+h_i-h_{i-1}=4v$. Also, $V_1=B_1$ so $|V_1|=|B_1|\leqslant h_1+v=4v$ and $V_j=T_j$ so $|V_j|=|T_j|\leqslant n-h_{j-1}+v\leqslant 4v$, completing the proof of (1).

We claim also that if x and y are incomparable vertices in V then there is an index i, $1\leqslant i\leqslant j$, such that x and y both belong to V_i. Indeed, as x and y are incomparable, there is an extension of $\vec{G}$ to a complete order in which x is the predecessor of y; say, x is the hth element and y is the $(h+1)$st. If we interchange x and y then we get a complete order extending $\vec{G}$ in which x is the $(h+1)$st element and y is the hth. In particular, $o_*(x)\leqslant h<o^*(x)$ and $o_*(y)\leqslant h<o^*(y)$. Let $1\leqslant i\leqslant j$ be such that $h_{i-1}<h\leqslant h_i$. Then $o_*(x)\leqslant h_i$, $o^*(x)>h_{i-1}$, $o_*(y)\leqslant h_i$ and $o^*(y)>h_{i-1}$ so $x,y\in V_i$.

Having used G to sort our elements in the first round, to complete the sorting in $r-1$ rounds, it suffices to sort each set V_i in $r-1$ rounds. That can be done with

$$\sum_{i=1}^{j}\mathrm{SORT}(|V_i|, r-1)\leqslant j\,\mathrm{SORT}(4v, r-1)$$

questions a round, implying the assertion of our lemma. □

Our next aim (Theorem 4) is to show that in a suitable space of random graphs, we can find an appropriate $(a; a_1, d_1; a_2, d_2)$-graph. This result can be proved in many ways; a part of the assertion will be deduced from the following lemma of Pippenger [18], which is essentially from [3]. For the sake of completeness we include its short and simple proof.

Lemma 3. *Let G be a graph whose complement does not contain a $K(b+1, b+1)$, and let $V_0 \subset V = V(G)$, $|V_0| = 5b$. Then there is a set $B \subset V$, $|B| \leqslant b$, such that if $X \subset V \setminus (V_0 \cup B)$ and $|X| \leqslant b$ then*

$$|\Gamma(X) \cap V_0| \geqslant 2\,|X|.$$

Proof. Let $\boldsymbol{B} = \{B \subset V \setminus V_0 : |B| \leqslant b, |\Gamma(B) \cap V_0| \leqslant 2\,|B| - 1\}$. If $\boldsymbol{B} = \emptyset$ then $B = \emptyset$ will do. Otherwise let B be a maximal element of $\boldsymbol{B}$. Since no edge joint B to $V_0 \setminus \Gamma(B)$ and $V_0 \setminus |\Gamma(B)| \leqslant 5b - (4b - 1) = b + 1$, we have $|B| \leqslant b$. Also, if $X \subset V \setminus (V_0 \cup B)$ and $1 \leqslant |X| \leqslant b$ then $X \cup B \notin \boldsymbol{B}$ so

$$|\Gamma(B \cup X) \cap V_0| \geqslant 2\,|B \cup X| = 2\,|B| + 2\,|X|,$$

implying

$$|\Gamma(X) \cap V_0| \geqslant 2\,|X| + 1. \quad \square$$

Theorem 4. *There is an n_0 such that the following assertion holds. Let $n \geqslant n_0$, $2 \leqslant r = r(n) \leqslant \log\log\log n$, $\alpha = 2^{r+1}$, $\mu = \frac{1}{4}(\log n)/\log\log n$, $m = n^{1-1/r}(\log n)^{2/r-1}(\log\log n)^{(r-1)/r}$, $a = (3/\alpha)m = 3.2^{-r-1}m$, $a_1 = 2^{-\mu}m$, $d_1 = (\log n)^{\frac{2}{3}}$, $a_2 = a/5$ and $d_2 = 2$. Then there is an $(a; a_1, d_1; a_2, d_2)$-graph of order n and size at most $\alpha(\log n)n^2/m$.*

Proof. Set $p = 2\alpha n^{1/r}(\log n)^{2-2/r}(\log\log n)^{-(r-1)/r}/n = 2\alpha(\log n)/m$ and consider the space $G(n, p)$ of random graphs. To prove the theorem, it suffices to show that if n is sufficiently large, with the threshold not depending on $r = r(n)$, then the probability that G_p is an $(a; a_i, d_i)$-graph is at least $\frac{5}{6}$ for $i = 1, 2$. Indeed, as the probability that the size of G_p is at most $\alpha(\log n)n^2/m = pn^2/2$ tends to $\frac{1}{2}$, this implies that if n is large enough then the probability that G_p will do the for the graph in our theorem is at least $\frac{1}{8}$, say. In fact, we shall show that as $n \to \infty$, *almost every* G_p is an $(a; a_1, d_1; a_2, d_2)$-graph.

Let us see first that *a.e.* G_p is an $(a; a_2, d_2)$-graph. By Lemma 3 is suffices to show that almost no $\bar{G}_p$ contains a $K(b+1, b+1)$, where $b = \lfloor a_2 \rfloor$. Let $X = S(G_p)$ be the number of $K(b+1, b+1)$ graphs in $\bar{G}_p$. Then

$$E(X) = \frac{1}{2}\binom{n}{b+1}\binom{n-b-1}{b+1}(1-p)^{b^2} \leqslant \left(\frac{en}{a_2}\right)^{a_2} e^{-pa_2^2}.$$

Since $5\alpha \leqslant (\log\log n)^{(r-1)/r}$ if n is large enough, $(en/a_2)^2 \leqslant n$ if n is large enough. Also, $pa_2 = \frac{6}{5}\log n$, so

$$E(X) \leqslant n^{-a_2/5} \leqslant n^{-n^{\frac{1}{2}}}$$

if n is sufficiently large.

Let us turn to the proof of the fact that *a.e.* G_p is an $(a; a_1, d_1)$-graph. Set $k = \lceil \log n \rceil$ and write P_0 for the probability that a fixed vertex x is joined to at

most k vertices in a fixed set of $\bar{a} = \lfloor a \rfloor$ vertices. Since $k < \frac{1}{3}ap$,

$$P_0 \leqslant 2\binom{a}{k} p^k (1-p)^{a-k} \leqslant 2\left(\frac{eap}{k}\right)^k e^{-ap+kp} \leqslant 3(6e)^{\lceil \log n \rceil} e^{-6\log n} < n^{-3}$$

if n is sufficiently large.

Let Ω_1 denote the event that G_p contains two disjoint sets of $\bar{a}$ vertices each, say A_1 and A_2, such that every vertex in A_1 is joined to at most k vertices in A_2. Clearly,

$$P(\Omega_1) \leqslant \binom{n}{\bar{a}}\binom{n-\bar{a}}{\bar{a}} P_0^{\bar{a}} \leqslant \left\{\left(\frac{en}{a}\right)^2 n^{-3}\right\}^{\bar{a}} < n^{-2} \tag{2}$$

if n is large enough.

We shall show that, on the other hand, the probability of the event Ω_2 that there are sets $X_1, X_2 \subset V(G_p)$ such that $|X_1| \leqslant S = \lfloor a_1 \rfloor$, $|X_2| \leqslant d_1 |X_1|$ and every vertex in X_1 is joined to at least k vertices in X_2 is also rather small. Since the subgraph spanned by $X_1 \cup X_2$ has at least $k|X_1|$ edges, we have the following rather crude estimate;

$$P(\Omega_2) \leqslant \sum_{s=1}^{\lfloor a_1 \rfloor} \binom{n}{s}\binom{n}{t}\binom{(s+t)^2/2}{ks} p^{ks}$$

where $t = \lfloor sd_1 \rfloor = \lfloor s(\log n)^{\frac{3}{4}} \rfloor$. Hence

$$P(\Omega_2) \leqslant 2\left(\frac{en}{a_1}\right)^{2d_1 a_1} \left(\frac{ed_1^2 a_1 p}{k}\right)^{k\lfloor a_1 \rfloor}$$

Note that

$$\frac{k/(ed_1^2 a_1 p) > 2^{\mu}}{(\log n)^2} > 2^{\mu/2}$$

and

$$\frac{3d_1 a_1 \log n < \mu k \lfloor a_1 \rfloor}{10}$$

so

$$P(\Omega_2) \leqslant 2^{-\mu k a_1/10} < n^{-2} \tag{3}$$

if n is large enough.

By inequalities (2) and (3) we have $P(\Omega_1 \cup \Omega_2) \leqslant 2n^{-2}$. Therefore the proof of the theorem will be complete if we show that if G_p is such that neither Ω_1 nor Ω_2 holds (i.e. $G_p \notin \Omega_1 \cup \Omega_2$) then G_p is an $(a; a_1, d_1)$-graph. Suppose then that $G_p \notin \Omega_1 \cup \Omega_2$ and $V_1 \subset V$ has at least a vertices. Let $W_1 = \{x \in V \setminus V : x$ is joined to at least a vertices in $V_1\}$. Then $|W_1| \geqslant |V \setminus V_1| - a$ since, Ω_1 holds. Suppose $X_1 \subset W_1$ and $|X_1| \leqslant a_1$. Let $X_2 = \Gamma(X_1) \cap V_1$. Then, since Ω_2 holds, $|X_2| > d_1 |X_1|$. □

We are ready to prove the main result of the paper.

Theorem 5. *There is an* n_1 *such that if* $n \geqslant n_1$ *and* $2 \leqslant r = r(n) \leqslant \log\log\log n$ *then*

$$SORT(n, r) \leqslant f(n, r),$$

where

$$f(n, r) = 2^{r+1} n^{1+1/r} (\log n)^{2-2/r} (\log\log n)^{-(r-1)/r}$$

Proof. Let $n \geqslant n_0$, α, μ, r, a_1, d_1, a_2, d_2 be as in Theorem 4 and let G be the $(a; a_1, d_1; a_2, d_2)$-graph of order n and size at most $f(n, r)$ whose existence is guaranteed by Theorem 4. Set

$$u = \lceil \log a_1/\log d_1 + \log(a_2/a_1)/\log d_2 \rceil + 1,$$

as in Lemma 2. Then $a_2/a_1 < 2^{\mu-2}$ so

$$u \leqslant \frac{3 \log a_1}{2 \log\log n} + \frac{\log(a_2/a_1)}{\log 2} + 2 < \frac{3}{2}\frac{\log n}{\log\log n} + \mu < 2\frac{\log n}{\log\log n} - 1.$$

This implies that

$$v = \lfloor 4(\log n/\log\log n)a \rfloor = \lfloor 3.2^{-r+1} n^{1-1/r} (\log n)^{2/r} (\log\log n)^{-1/r} \rfloor$$

can be shown in is Lemma 2, so

$$\mathrm{SORT}(n, r) \leqslant \max\left\{f(n, r), \frac{n}{2} v\, \mathrm{SORT}(4v, r-1)\right\}.$$

Consequently,

$$\mathrm{SORT}(n, r) \leqslant f(n, r)$$

if

$$\frac{n}{2} v\, \mathrm{SORT}(4v, r-1) \leqslant f(n, r).$$

Since $\mathrm{SORT}(4v, a) = (\tfrac{4v}{2} < 8v^2$ and

$$4nv \leqslant 6n^{\frac{3}{2}} (\log n)(\log\log n)^{-\frac{1}{2}} < f(n, 2) = 8n^{\frac{3}{2}} (\log n)(\log\log n)^{-\frac{1}{2}},$$

this implies that if $n \geqslant n_0$ then $\mathrm{SORT}(n, 2) < f(n, 2)$.

We shall prove by induction on r that if $2 \leqslant r \leqslant \log\log\log n$ and $n^{1/r} > n_0$ then $\mathrm{SORT}(n, r) \leqslant f(n, r)$. This implies the theorem since it gives that if $n > n_0^{n_0}$, say, then $\mathrm{SORT}(n, r) \leqslant f(n, r)$ for all r, $2 \leqslant r \leqslant \log\log\log n$. As the case $r = 2$ has been proved, let us assume that $3 \leqslant r \leqslant \log\log n$ and the assertion has been proved for smaller values of n. Note that $v > n^{(r-1)/r}$ so $v^{1/(r-1)} > n_0$. Hence, by the induction hypothesis, $\mathrm{SORT}(4v, r-1) \leqslant f(4v, r-1)$. Therefore

$$\begin{aligned}
\frac{n}{2v} \mathrm{SORT}(4v, r-1) &\leqslant \frac{n}{2} v 2^r (4v)^{1+1/(r-1)} (\log 4v)^{2-2/(r-1)} (\log\log 4v)^{-(r-2)/(r-1)} \\
&\leqslant 2^{r+1} n 4^{1/(r-1)} v^{1/(r-1)} (\log n)^{2-2/(r-1)} (\log\log n)^{-(r-2)/(r-1)} \\
&\leqslant 2^{r+1} n^{1+1/r} (\log n)^{2-2/r} (\log\log n)^{-(r-1)/r} = f(n, r),
\end{aligned}$$

proving the induction step. □

In conclusion, let us note that the networks we use are essentially independent of the answers we get, i.e. they hardly depend on the order of the elements. To be precise, the sorting algorithm implied by the proof of Theorem 5 makes use of r graphs, say $G_1, G_2, \ldots, G_r$, depending only on n and r, such that the first round questions are asked according to G_1, then V is covered by some sets $V_1, V_2, \ldots, V_s$ of order $|G_2|$ and in the next round each V_i is sorted by G_2. Then each V_i is covered by sets $V_{i1}, V_{i2}, \ldots, V_{it}$ of order $|G_3|$ and each V_{ij} is sorted by G_3, etc. In the rth round we compare all pairs of elements belonging to some set $V_{i_1 i_2 \cdots i_{r-1}}$.

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ON EDGE-HAMILTONIAN PROPERTY OF CAYLEY GRAPHS

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Let G be a group generated by X. A *Cayley graph over* G is defined as a graph $G(X)$ whose vertex set is G and whose edge set consists of all unordered pairs $[a, b]$ with $a, b \in G$ and $a^{-1}b \in X \cup X^{-1}$, where X^{-1} denotes the set $\{x^{-1} \mid x \in X\}$. When X is a minimal generating set or each element of X is of even order, it can be shown that $G(X)$ is Hamiltonian iff it is edge-Hamiltonian. Hence every Cayley graph of order a power of 2 is edge-Hamiltonian.

1. Introduction

Throughout this paper, we shall only consider finite undirected simple graphs. For each such graph Γ, we shall denote the vertex set of Γ by $V(\Gamma)$ and the edge set by $E(\Gamma)$. The letter G will always denote a finite group generated by a set X where the identity ι of G is not in X. A *Cayley graph over* G is defined as a graph $G(X)$ whose vertex set is G and whose edge set consists of all unordered pairs $[a, b]$ with $a, b \in G$ and $a^{-1}b \in X \cup X^{-1}$ ($X^{-1} = \{x^{-1} \mid x \in X\}$). In other words, two vertices a, b of $V(G)$ are adjacent iff $ax = b$ or $ax^{-1} = b$ for some x in X. We shall call the corresponding edge $[a, b]$ an *x-edge* and a, b the *ends* of $[a, b]$. It is easy to see that each element a of G gives rise to an automorphism Θ_a of $G(X)$ where $\Theta_a : x \to ax$. Further, the mapping $a \to \Theta_a$ is an embedding of G into the automorphism group of $G(X)$. Because of this, it is clear that the graph $G(X)$ is vertex-transitive in the sense that for any two vertices a, b of $G(X)$, there exists an automorphism (namely, Θ_c where $c = ba^{-1}$) of $G(X)$ which maps a to b. In fact, the converse is almost true, for there are only four known examples of connected vertex-transitive graphs which are not Cayley graphs.

Among the many conjectures which intrigue combinatorists nowadays, the following conjecture of Lovasz is undoubtedly one that attracts much attention.

Conjecture 1. [Lovasz] Every connected vertex-transitive graph has a Hamilton path.

As Cayley graphs form a special class of vertex-transitive graphs, the above conjecture naturally leads to the following more specific conjecture:

Conjecture 2. Every Cayley graph has a Hamilton cycle.

Though much effort has been made in order to verify the conjecture, yet up to date, it is known that the conjecture is only true for Cayley graphs over some very special classes of groups. For instance, it is fairly easy to show that Cayley graphs over Abelian groups are always Hamiltonian. However, for Cayley graphs over non-Abelian groups, the conjecture is far from being solved. Chen and Quimpo prove in [2] that Cayley graphs over groups of order pq, where p, q are primes, are Hamiltonian and in [3] that Cayley graphs over Hamiltonian groups (i.e. non Abelian groups in which every subgroup is normal) are Hamiltonian. The best known result so far is perhaps the one proved by Keating and Witte in [4], which states that if G is a group whose commutator subgroup is a cyclic group of prime power order, then any Cayley graph over G is Hamiltonian. Witte [5] also showed recently that every Cayley graph of prime power order is Hamiltonian. Furthermore, for most of these known results, it can be shown that the corresponding Hamiltonian Cayley graphs are in fact *edge-Hamiltonian*, in the sense that every edge lies on a Hamilton cycle. This in turn leads to another interesting conjecture:

Conjecture 3. Every Hamiltonian Cayley graph is edge-Hamiltonian.

The main objective of this paper is to give a partial solution to this conjecture. Among others, we shall show that when each element of X is of even order, $G(X)$ is Hamiltonian iff it is edge-Hamiltonian. Hence Cayley graph over a group of order a power of 2 is always edge-Hamiltonian.

2. Basic lemmas

The following lemmas will be useful in the sequel.

Lemma 1. *A Cayley graph $G(X)$ is edge-Hamiltonian iff for each x in X, $G(X)$ contains a Hamilton cycle containing an x-edge.*

Proof. Apparently if $G(X)$ is Hamiltonian, then the given condition holds. Conversely, assume that for each x in X, $G(X)$ contains a Hamilton cycle with an x-edge. To prove that $G(X)$ is edge-Hamiltonian, let $e = [a, b]$ be any edge of $G(X)$ with $a^{-1}b = x \in X \cup X^{-1}$. Without loss of generality, we may assume that $x \in X$. By the given condition, G has a Hamilton cycle C containing an edge $e' = [c, d]$ with $c^{-1}d = x$. Now consider the mapping $\Theta_g : y \to gy$ where $g = ac^{-1}$. Then Θ_g is an automorphism of $G(X)$ and so $\Theta_g(C)$ is also a Hamilton cycle of $G(X)$. However, $\Theta_g(c) = gc = a$ and $\Theta_g(d) = gd = gcc^{-1}d = gcx = ac^{-1}ca^{-1}b = b$. Hence the edge $[a, b]$ lies on the Hamilton cycle $\Theta_g(C)$ which proves that $G(X)$ is edge-Hamiltonian, as required. □

Lemma 2. *Let C be a Hamilton cycle of a Cayley graph $G(X)$. Let $Y = \{a^{-1}b \mid [a, b] \in E(C)\}$. Then Y generates G.*

Proof. Let the Hamilton cycle C be $\langle v_0, v_1, \ldots, v_n \rangle$, where n is the order of G and $v_0 = v_n = \iota$, the identity of G. Let $v_i^{-1}v_{i+1} = x_i$ where $i = 0, 1, \ldots, n-1$. Then the set $Y' = \{x_i \mid i = 0, 1, \ldots, n-1\}$ is contained by Y. Moreover, as each v_i is the product of $x_0, x_1, \ldots, x_{i-1}$, we see that Y' generates G and so Y also generates G, as required. $\square$

Let Γ be a cubic graph. By a *perfect matching partition* (*PMP*) of Γ, we mean an unordered triple $\bar{P} = [P_1, P_2, P_3]$ where the P_i's are edge-disjoint perfect matchings of Γ. Apparently, they form a partition of $E(\Gamma)$ into perfect matchings. Moreover, by an *even cycle partition* (*ECP*) of Γ, we mean a set $\bar{C} = \{C_1, \ldots, C_k\}$ of vertex-disjoint cycles of Γ whose union is $V(\Gamma)$. Hence $\bar{C}$ is a partition of $V(\Gamma)$ into even cycles. It is easy to see that each (PMP) $\bar{P} = [P_1, P_2, P_3]$ gives rise to exactly three (ECP)'s, namely, $\bar{C}_1 = P_1 \cup P_2$, $\bar{C}_2 = P_2 \cup P_3$ and $\bar{C}_3 = P_3 \cup P_1$. On the other hand, for each (ECP) $\bar{C} = \{C_1, \ldots, C_k\}$ there are exactly 2^{k-1} (PMP)'s which give rise to $\bar{C}$, because each even cycle has two perfect matchings and the set of all edges not in $C_1, \ldots, C_k$ also form a perfect matching of Γ. With this in mind, we can now establish the following:

Lemma 3. *Every edge in a cubic graph lies on an even number of Hamilton cycles.*

Proof. Let Γ be any cubic graph and e an edge of Γ. Consider the following table each row of which corresponds to a (PMP) and each column of which corresponds to an (ECP). As each (PMP) $\bar{P} = [P_1, P_2, P_3]$ gives rise to three (ECP)'s $\bar{C}_1 = P_1 \cup P_2$, $\bar{C}_2 = P_2 \cup P_3$ and $\bar{C}_3 = P_3 \cup P_1$, we shall complete the table by filling in the three (ECP)'s $P_1 \cup P_2$, $P_2 \cup P_3$, $P_3 \cup P_1$ in the row corresponding to $\bar{P} = [P_1, P_2, P_3]$ and in the columns corresponding to $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$ respectively. All other entries in this row will be filled up by empty sets.

(ECP) / (PMP)	$\cdots$	$\bar{C}_1$	$\cdots$	$\bar{C}_2$	$\cdots$	$\bar{C}_3$	$\cdots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$	
$\bar{P}$	$\cdots$	$P_1 \cup P_2$	$\cdots$	$P_2 \cup P_3$	$\cdots$	$P_3 \cup P_1$	$\cdots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$	

Now we shall count the number N of occurrences of the edge e in the whole table. Evidently, each row contains the edge e exactly twice. Hence, $N = 2r$ where r is the number of rows, or the number of (PMP)'s of Γ. On the other hand, if we count over a column corresponding to an (ECP) $\bar{C} = \{C_1, \ldots, C_k\}$,

as this (ECP) occurs at exactly 2^{k-1} locations in the column, the number of occurrences of e in the column is either 0 or 2^{k-1} which is always even for $k>1$ and is equal to 1 for $k=1$, in which case $\bar{C}$ consists of exactly one Hamilton cycle. Hence the number N of occurrences of e in the table is equal to the sum of the number n_1 of Hamilton cycles containing e and an even number n_2. That is, $N=n_1+n_2=2r$. Therefore n_1 must be even, as required. □

Lemma 4. *Every Hamiltonian cubic graph contains at least* 3 *Hamilton cycles.*

Proof. Let Γ be a Hamiltonian cubic graph. By Lemma 3, G contains at least two Hamilton cycles C_1 and C_2, say. There must exist an edge e of Γ which is in C_1 but not in C_2. Again, by Lemma 3, there is a Hamilton cycle C_3 other than C_1 that also contains e. Evidently C_3 is different from C_2 and so we have at least three Hamilton cycles C_1, C_2 and C_3 in Γ. □

To end this section, we would like to raise the following question:

Problem. Does every regular Hamiltonian graph other than a cycle contains more than one Hamilton cycles? In particular, does every 4-regular Hamiltonian graph contain more than one Hamilton cycles?

3. Edge Hamiltonian property of Cayley graphs

With the basic lemmas established in the previous section, we are now in a position to prove the following theorems.

Theorem 5. *Let X be a minimal generating set of the group G. Then $G(X)$ is Hamiltonian iff it is edge-Hamiltonian.*

Proof. The sufficiency is clear. To prove the necessity, assume that $G(X)$ is Hamiltonian. Let C be any Hamilton cycle of G. It follows from Lemma 2 that the set $A=\{a^{-1}b \mid [a,b]\in E(C)\}$ generates G. As this is a subset of $X\cup X^{-1}$, by minimality of X, we must have $X\subset A$. Hence for each x in X, there exists an edge $[a,b]$ of C with $a^{-1}b=x$. Thus C contains an x-edge. Hence by Lemma 1, $G(X)$ is edge-Hamiltonian. □

Theorem 6. *Let $G(X)$ by a Cayley graph where each element of X is of even order. Then $G(X)$ is Hamiltonian iff $G(X)$ is edge-Hamiltonian.*

Proof. Again the sufficiently is clear. To prove the necessity, let x be any element of X. We shall show that $G(X)$ contains a Hamilton cycle with an x-edge. Let C be any Hamilton cycle of $G(X)$. If C contains an x-edge, then we are through. Assume therefore that C does not contain an x-edge.

If the order of x is 2, then C together with all the x-edges of $G(X)$ form a cubic Hamiltonian graph Γ in which all the x-edges are 'chords' of C. By Lemma 4, Γ must contain a Hamilton cycle C' other than C. Thus, Γ (which is also a Hamilton cycle of $G(X)$) must contain a chord which is an x-edge that we require.

Finally, consider the case when the order of x is $2k$ where k is an integer greater than 1. In this case, the set of all x-edges of $G(X)$ forms an (ECP) $\bar{D}$ of $G(X)$. The Hamilton cycle C together with $\bar{D}$ gives rise to a subgraph Ω of $G(X)$ which is a 4-regular Hamiltonian graph. As every even cycle contains a perfect matching, Ω contains a perfect matching P whose edges are chosen from cycles of $\bar{D}$. Hence C together with P form a subgraph Ω' of Ω which is a cubic Hamiltonian graph. Now as in the first part of the proof, Ω' contains another Hamilton cycle with an x-edge, which is also a Hamilton cycle of $G(X)$. As the element x of X is arbitrarily chosen, the proof that $G(X)$ is edge-Hamiltonian follows from Lemma 1. $\square$

Combining this and the fact that Cayley graphs of prime power order are Hamiltonian (Witte [5]), we have the following immediate consequence:

Theorem 7. *Every Cayley graph of order* 2^k $(k = 2, 3, \ldots)$ *is edge-Hamiltonian.*

To end this paper, we wish to point out that from the proof of Theorem 6, it is clear that if the question to the problem raised at the end of the previous section is in the affirmative, then Conjecture 3 will also be established.

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GALLAI THEOREMS FOR GRAPHS, HYPERGRAPHS, AND SET SYSTEMS

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1. Introduction

In 1959 Gallai [4] presented his now classical theorem, involving the vertex covering number α_0, the vertex independence number β_0, the edge covering number α_1, and the maximum matching (or edge independence) number β_1.

Theorem 1 (Gallai). *For any nontrivial, connected graph $G = (V, E)$ with p vertices,*

I. $\alpha_0 + \beta_0 = p$
II. $\alpha_1 + \beta_1 = p$.

Since then quite a large number of similar results and generalizations of this theorem have been obtained, which we will call 'Gallai Theorems'. A typical Gallai theorem has the form:

$$\alpha + \beta = p,$$

where α and β are numerical maximum or minimum functions of some type defined on the class of connected graphs and p denotes the number of vertices in a graph.

This paper is an attempt to collect and unify results of this type. In particular, we present two general theorems which encompass nearly all of the existing Gallai theorems. The first theorem is based on hereditary properties of set systems, while the second is based on partitions of vertices into subgraphs having

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treelike properties. We also present a variety of new Gallai theorems (one of which is not a corollary of either of the two above mentioned generalizations), as well as a number of other new results.

2. Gallai theorems for hereditary set systems

In this section we will prove a general theorem concerning hereditary properties of sets, from which one can obtain as corollaries a variety of Gallai theorems, including the original Gallai Theorem given above.

Let S be a finite set of n elements and let P be a *hereditary property* of the subsets of S, i.e. P is a function f from the power set of S to $\{0, 1\}$ such that $f(X)=1$ and $X' \subseteq X$ implies $f(X')=1$. If $f(X)=1$ we say that subset X *has property* P and that X is a *P-set*; if $f(X)=0$ then X is called a *P'-set.*

A transversal of a family $\mathcal{F}$ of sets is a set T such that $|T \cap F| \geqslant 1$ for each $F \in \mathcal{F}$.

Theorem 2. *Let $X \subseteq S$. Then X is a maximal P-set for some hereditary property P if and only if $S-X$ is a minimal transversal of the class of all P'-sets.*

Proof. Let X be a maximal P-set and Y any P'-set. Then $Y \cap (S-X) \neq \emptyset$, otherwise $Y \subseteq X$. But this would imply that Y is a P-set since P is a hereditary property. Hence, $S-X$ is a P'-set transversal. It remains to show that $S-X$ is a minimal transversal.

Suppose not. Then for some $u \in S-X$, $Z=S-X-\{u\}$ is a P'-set transversal. But $X \cup \{u\}$ does not intersect Z. Hence $X \cup \{u\}$ must be a P-set, contradicting the maximality of X.

Conversely, suppose that $S-X$ is a minimal transversal of the P'-sets. Then, since $S-X$ does not intersect X, X must be a P-set. We must show that X is a maximal P-set. Suppose for some $u \in S-X$, $X \cup \{u\}$ is a P-set. Then since there are no P'-sets contained in $X \cup \{u\}$, $S-X-\{u\}$ intersects all P'-sets, contradicting the minimality of $S-X$. Hence X is a maximal P-set, as required. □

Note that X is a minimal P'-set transversal if and only if X is a minimal transversal of the *minimal* P'-sets.

Suppose now that there is a positive weight $w(s)$ associated with each element $s \in S$. Define $\alpha^+(P)(\alpha^-(P))$ be the largest (smallest) sum of the weights of the elements in a minimal P'-set transversal; similarly, let $\beta^+(P)(\beta^-(P))$ be the largest (smallest) sum of the weights of the elements in a maximal P-set. The following weighted Gallai theorems follow immediately from Theorem 2.

Corollary 2a. *For any hereditary property P of a set system,*

(i) $\alpha^-(P)+\beta^+(P)=\sum_{s \in S} w(s)$

(ii) $\alpha^+(P)+\beta^-(P)=\sum_{s \in S} w(s)$

In stating the following corollaries of Theorem 2, we will assume that all elements in S have unit weight, so that the sum of the weights of the elements in a set X equals the cardinality of X. We note, however, that each of the following corollaries can be generalized to the arbitrary weighted case.

For example, Cockayne and Giles [2] noticed Corollary 2a(i) in the unweighted case.

Hedetniemi [7] also obtained special cases of Corollary 2a(i) when the basic set S is either the vertex set $V(G)$ or the edge set $E(G)$ of a graph $G=(V, E)$.

Part I of Gallai's theorem can be obtained from the unweighted version of Corollary 2a(i). We let $S=V(G)$ and we say that a set $X \subseteq S$ has property P if and only if $\langle X \rangle$ the subgraph induced by X is totally disconnected (i.e. X is an independent set of vertices). In this case, minimal P'-set transversals are minimal vertex sets which cover all minimal non-independent vertex sets, i.e. pairs of adjacent vertices. Thus, minimal P'-transversals are minimal vertex covers (sets of vertices which cover all edges). In this context, $\beta^-(P)=i(G)$, the independent domination number (equivalently, the smallest number of vertices in a maximal independent set), while $\alpha^+(P)=\alpha_0^+(G)$ is the maximum number of vertices in a minimal vertex cover.

Corollary 2b. *For any nontrivial connected graph G with p vertices,*

(i) (*Gallai* [4]) $\quad \alpha_0+\beta_0=p$

(ii) (*McFall, Nowakowski* [14]) $\quad \alpha_0^+ + \beta_0^- = p.$

Let set S be the edge set $E(G)$ of a graph G, where $|E(G)|=q$. Let a subset $X \subseteq S$ of edges have property P if and only if X is independent, i.e. X is a matching. A minimal P'-set in this case is a pair of adjacent edges. In this context, $\beta^+(P)=\beta_1(G)$, the matching number of G; while $\beta^-(P)=\beta_1^-(G)$ equals the smallest number of edges in a maximal matching (i.e. a minimaximal matching). The parameter $\alpha^-(P)$ can in this case be seen to equal $\alpha_0(L(G))$, the vertex covering number of the line graph $L(G)$ of G. We have:

Corollary 2c. *For any nontrivial connected graph G with p vertices and q edges,*

(i) (*Hedetniemi* [7] $\quad \alpha_0(L(G))+\beta_1(G)=q$

(ii) $\quad \alpha_0^+(L(G))+\beta_1^-(G)=q.$

Many Gallai theorems may be obtained by considering a class $\mathscr{G}$ of forbidden subgraphs, letting $S=V(G)$ (or $E(G)$) and saying that a set $X \subseteq S$ has property P if and only if the induced subgraph $\langle X \rangle$ contains no member of $\mathscr{G}$.

For example, let $\mathscr{G}=\{K_3\}$ and $S=V(G)$.

Corollary 2d. *Let α_3 denote the minimum number of vertices covering all the triangles of a graph G with p vertices, and let β_3 denote the maximum number of vertices in a set S such that $\langle S \rangle$ contains no triangles. Then*

$$\alpha_3+\beta_3=p.$$

Our next application for Theorem 2 provides another result on matching and introduces a new graph theory parameter called the *matchability number of a graph*. Let $S = V(G)$ and let a set $X = \{x_1, x_2, \ldots, x_t\} \subseteq V$ have property P if and only if X is *matchable*, i.e. there exists a set of t independent edges $(x_i, f(x_i))$, $i = 1, 2, \ldots, t$. Let $\alpha_m^+(\alpha_m^-)$ denote the maximum (minimum) number of vertices in a minimal transversal of non-matchable sets, and let $\beta^+(\beta^-)$ denote the maximum (minimum) number of vertices in a maximal matchable set. Notice that $\beta^+(P) = \beta_1(G)$ (the matching number).

Corollary 2e. *For any nontrivial, connected graph G with p vertices,*
(i) $\alpha_m^- + \beta_1 = p$
(ii) $\alpha_m^+ + \beta^- = p$.

From Corollary 2e(i) and Gallai's Theorem, Part II, we may immediately conclude

Corollary 2f. *For any nontrivial connected graph G.*

$$\alpha_m^-(G) = \alpha_1(G).$$

The *matchability number of a graph*, $\beta^-(G)$, i.e. the smallest cardinality of a maximal matchable set of vertices, appears to be a new parameter. It is in general neither equal to $\beta_1^-(G)$, the smallest number of edges in a maximal matching nor equal to $\beta_1(G)$, the matching number. This can be seen from the graphs G_1 and G_2 in Fig. 1. It is easy to see, in fact, that for any graph G,

$$\gamma'(G) = \beta_1^-(G) \leq \beta^-(G) \leq \beta^+(G) = \beta_1(G)$$

where $\gamma'(G)$ is the edge domination number of G. A graph G_1 with $\beta_1^- = 2 < \beta^- = 3$ and G_2 with $\beta^- = 2 < \beta_1 = 3$, where a $\beta_1^-(G_1)$-set is $\{(2,3),(4,6)\}$; and a $\beta^-(G)$-set is $\{2,3,4\}$; a $\beta^-(G_2)$-set is $\{3,4\}$ and a $\beta_1(G_2)$-set is $\{(1,2),(3,4),(5,6)\}$.

Next, we let $S = V(G)$ and let $X \subseteq S$ have property P if and only if X contains no closed neighborhood of G. We observe that a minimal transversal of the P'-sets is a minimal transversal of the set of closed neighborhoods, i.e. a minimal dominating set of G. The cardinalities of a largest and a smallest minimal dominating set are denoted $\Gamma(G)$ and $\gamma(G)$, respectively. Let $\beta_c^+(G)$ and $\beta_c^-(G)$ denote the cardinality of largest and smallest P-sets, respectively.

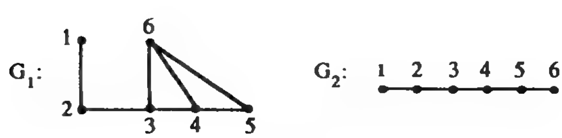

Fig. 1.

Corollary 2g. *For any nontrivial connected graph G with p vertices,*
(i) $\gamma + \beta_c^+ = p$
(ii) $\Gamma + \beta_c^- = p$.

Let $\varepsilon(G)$ denote the maximum number of pendant edges in a spanning forest of G. In [15] Nieminen proved

Theorem 3 (Nieminen). *For any non-trivial connected graph G with p vertices,*

$$\gamma + \varepsilon = p.$$

In [8] Hedetniemi observed a duality between Nieminen's result and Part II of Gallai's theorem as follows. A spanning star partition is a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of V such that for $1 \leq i \leq k$, $\langle V_i \rangle$ contains a *non-trivial* spanning star. Define $\beta_*^+(\beta_*^-)$ to equal the maximum (minimum) order of a spanning star partition of G. Also let $\alpha_1^+(\alpha_1^-)$ equal the maximum (minimum) order of minimal edge cover (a set of edges which covers all vertices of G).

Theorem 3'. *For any nontrivial connected graph G with p vertices,*
(i) (Hedetniemi) $\alpha_1^- + \beta_*^+ = p = \alpha_1 + \beta_1$ (*Gallai II*)
(ii) (Hedetniemi) $\alpha_1^+ + \beta_*^- = p = \varepsilon + \gamma$ (*Nieminen*).

Thus we conclude from Corollary 2g(i) and Theorems 2 and 3 that

Corollary 3a. *for any nontrivial connected graph G,*

$$\beta_c^+ = \varepsilon = \alpha_1^+.$$

Similar results can be obtained when closed neighborhoods are replaced by open neighborhoods. In this case minimial P'-set transversals are minimal total dominating sets (cf. Cockayne, et al. [3]).

As our last illustration of Theorem 2, we let $S = V(H)$, the vertex set of a hypergraph H with edge set $\mathscr{E}$. For a given positive integer k, we say that a set X of S has property P if and only if $|e \cap X| \leq k$ for each $e \in \mathscr{E}$. Minimal P'-transversals are minimal transversals of the sets of vertices which contain at least $k+1$ vertices of some edge. The results in Corollary 2a(i) and (ii) in this context reduce to Corollaries 2b(i) and (ii) when H is a graph and $k = 1$.

Finally, we note that one can prove a Boolean dual of Theorem 2 which may have interesting special cases. Let Q be an *expanding* property on the subsets of a given set S, i.e. Q is a function f from the power set of S to $\{0, 1\}$ such that $f(X) = 1$ and $X' \supseteq X$ implies $f(X') = 1$.

Theorem 2'. *A set $X \subseteq S$ is a minimal Q-set if and only if $S - X$ is a maximal set whose union with any Q'-set is not S.*

3. Gallai theorems from spanning forests

An interesting variety of Gallai theorems can be obtained from a very elementary observation about spanning forests of a connected graph G. Let F be a spanning forest of a graph G with p vertices, let $e(F)$ denote the number of edges in F and $t(F)$ denote the number of connected components of F (i.e. the number of trees in the forest).

Proposition 4. *For any graph G with p vertices and any spanning forest F of G,*

$$t(F)+e(F)=p.$$

Next consider the class of all spanning forests satisfying some property P; let such a forest be called a P-forest. Let $t(P)$ denote the minimum number of trees in a (spanning) P-forest, and let $e(P)$ denote the maximum number of edges in a P-forest.

Corollary 4a. *For any graph G with p vertices and any property P of a spanning forest*

$$t(P)+e(P)=p.$$

A number of properties P of spanning forests lead to interesting Gallai theorems.

Let P_1 denote the property that every tree in a forest has at most two vertices (equivalently, has diameter $\leqslant 1$). Then it can be seen that $t(P_1)=\alpha_1(G)$, the vertex covering number, and $e(P_1)=\beta_1(G)$, the matching number.

Corollary 4b. *For any connected graph G with p vertices,*

$$t(P_1)+e(P_1)=\alpha_1+\beta_1=p \ (\textit{Gallai, II})$$

Next, let P_2 denote the property that every tree in a forest F has diameter $\leqslant 2$. In this case every tree in F is a star ($K_{1,n}$) and it can be seen that

$t(P_2)=\gamma$, the domination number, and
$e(P_2)=\varepsilon$, the pendant edge number

(the maximum number of pendant edges in a spanning forest).

Corollary 4c. *For any connected graph G with p vertices,*

$$t(P_2)+e(P_2)=\gamma+\varepsilon=p \ (\textit{Nieminen}).$$

One can observe from Corollary 4c, Corollary 2g, and Theorem 3 that there are four equivalent definitions of the domination number of a graph $G=(V, E)$:

(i) the minimum number of vertices in a set D such that every vertex in $V-D$ is adjacent to at least one vertex in D;

(ii) the minimum number of vertices in a transversal of the closed neighborhoods of G;
(iii) the minimum order of a spanning star partition of the vertices of G (note; this does not allow isolated vertices);
(iv) the minimum number of trees in a spanning forest in which every tree has diameter $\leqslant 2$ (note: this allows isolated vertices).

Finally, let P_* denote the property that every tree in a forest is a path. In this case $t(P_*)$ equals the minimum number of paths whose union is $V(G)$, and $e(P_*)$ is the maximum number of edges in a path decomposition of $V(G)$.

Corollary 4d. *For any graph G with p vertices, $t(P_*)+e(P_*)=p$.*

Notice that $t(P_*)=1$ if and only if G has a Hamiltonian path.

4. Other Gallai theorems

Additional Gallai theorems can be found in the literature which involve a wide variety of graphical parameters. In this section we mention a few of these. We also include a new theorem, which apparently is not a corollary of any of the previous results.

Let $\gamma_c(G)$, the *connected domination number* of G, equal the minimum number of vertices in a dominating set D such that $\langle D\rangle$ is a connected subgraph. Let $\varepsilon_T(G)$ equal the maximum number of pendant edges in a spanning tree of G.

Theorem 5 (Hedetniemi and Laskar [9]). *For a connected graph G with p vertices,*

$$\gamma_c(G)+\varepsilon_T(G)=p.$$

Let $\lambda(G)$, the *edge connectivity* of G, be the minimum number of edges in a set S, such that $G-S$ is disconnected or K_1. Definite $\beta_\lambda(G)$ to be the maximum number of edges in a set S that $\langle S\rangle$ is disconnected.

Theorem 6 (Hedetniemi [6]). *For a connected graph G with q edges,*

$$\lambda(G)+\beta_\lambda(G)=q.$$

This result follows from Corollary 2a.

Let $\lambda^+(G)$ denote the maximum cardinality of a minimal set of edges S, such that $G-S$ is disconnected or K_1, and let $\beta_\lambda^-(G)$ denoted the minimum number of edges in a maximal set of edges S such that $\langle S\rangle$ is disconnected. Then the following theorem also results from Corollary 2a.

Theorem 7 (Peters et al. [16]). *For any non-trivial connected graph G with q edges,*

$$\lambda^+(G)+\beta_\lambda^-(G)=q$$

Similar results hold for vertex-connectivity. Let $\kappa(G)$ denote the *vertex connectivity* of G, i.e. minimum number of vertices in a set S, such that $G-S$ is either disconnected or K_1. Let $\beta_\kappa(G)$ denote the maximum number of vertices in a set S such that $\langle S\rangle$ is disconnected or K_1.

Theorem 8 (Hedetniemi [6]). *For a connected graph G with p vertices,*

$$\kappa(G)+\beta_\kappa(G)=p.$$

If we replace $\kappa(G)$ by $\kappa^+(G)$ and $\beta_\kappa(G)$ by $\beta_\kappa^-(G)$ in the above theorem, we obtain another Gallai theorem, where $\kappa^+(G)$ is the maximum number of vertices in a minimal set S, such that $V-S$ is disconnected or K_1 and $\beta_\kappa^-(G)$ is the minimum number of vertices in a maximal set S, such that $\langle S\rangle$ is disconnected.

Theorem 9 (Hare, Laskar, Peters [5]). *For a connected graph G with p vertices,*

$$\kappa^+(G)+\beta_\kappa^-(G)=p.$$

We now introduce two new parameters, $\alpha_{1k}(G)$ *and* $\beta_{1k}(G)$ of a graph G and prove a Gallai theorem involving these parameters, which apparently is not a corollary of any of the previous results. As a matter of fact, if $k=1$, then Gallai theorem II results.

Let $k\leqslant\delta$, the minimum degree of G. Definite $\alpha_{1k}(G)$ to be the minimum number of edges in an edge set $X\subseteq E$ such that, for every $v\in V(G)$, $\deg v$ in $\langle X\rangle$ is at least k. Define $\beta_{1k}(G)$ to be the maximum number of edges in an edge set $X\subseteq E$ such that, for every $v\in V(G)$, $\deg v$ in $\langle X\rangle$ is at most k. Note that for $k=1$, $\alpha_{11}(G)=\alpha_1(G)$ is the edge-covering number and $\beta_{11}(G)=\beta_1(G)$ is the matching number of G.

Theorem 10. *For a connected graph G with p vertices and $k\leqslant\delta$,*

$$\alpha_{1k}(G)+\beta_{1k}(G)=kp.$$

Proof. Let X be an α_{1k}-set, i.e. X is a set of $\alpha_{1k}(G)$ edges such that for each $v\in V(G)$, $\deg v$ in $\langle X\rangle$ is at least k.

Let $A=\{v\mid \deg v \text{ in } \langle X\rangle=k\}$
$B=\{v\mid \deg v \text{ in } \langle X\rangle>k\}$.

Let $A=\{v_1,v_2,\ldots,v_s\}$ and $B=\{u_1,u_2,\ldots,u_t\}$. Suppose the degree of

$u_i \in B$ in $\langle X \rangle$ is $k+\lambda_i$, $i=1,2,\ldots,t$. Note that each $\lambda_i > 0$. Due to the minimality of $\langle X \rangle$, it follows that B is independent. Let Y denote the set of edges obtained from X by deleting λ_i edges from each $u_i \in B$, $i=1,2,\ldots,t$. Then for each $v_i \in A$, $\deg v_i$ in $\langle Y \rangle$ is at most k, and for each $u_i \in B$ $\deg u_i$ in $\langle Y \rangle$ is exactly k. In other words, for each $v \in V(G) \deg v$ in $\langle Y \rangle$ is at most k, and hence, $\beta_{1k}(G) \geq |Y|$. Counting the degrees of each $v \in V(G)$ in $\langle X \rangle$ and $\langle Y \rangle$ we get

$$\begin{aligned} sk + \sum_{i=1}^{t} (k+\lambda_i) + tk + ks - \sum_{i=1}^{t} \lambda_i & \\ &= 2k(t+s) \\ &= 2kp. \end{aligned}$$

Hence, the sum of the numbers of edges in X and Y is kp and we have

$$\alpha_{1k}(G) + \beta_{1k}(G) \geq |X| + |Y| = kp.$$

To show that $\alpha_{1k}(G) + \beta_{1k}(G) \leq kp$, we let X be a β_{1k}-set, i.e. a set of $\beta_{1k}(G)$ edges such that for each $v \in V(G)$, $\deg v$ in $\langle X \rangle$ is at most k. Let

$A = \{v \mid \deg v \text{ in } \langle X \rangle = k\}$

$B = \{v \mid \deg v \text{ in } \langle X \rangle < k\}$.

Let $A = \{v_1, v_2, \ldots, v_s\}$ and $B = \{u_1, u_2, \ldots, u_t\}$. Suppose $\deg u_i$ in $\langle X \rangle$ is $k - \lambda_i$, $i=1,2,\ldots,t$. Note that $\lambda_i > 0$. Due to the maximality of X, it follows that if $u_iu_j \in X$, then $u_iu_j \in E$. Now, as before we construct a set Y of edges from X so that for each $v \in V(G)$, $\deg v$ in $\langle Y \rangle$ is at least k. This can be done by adding λ_i edges to each vertex $u_i \in B$, $i=1,2,\ldots,t$, and these edges join vertices of B with A. Note that $|Y| \geq \alpha_{1k}(G)$. Counting the degrees of vertices in $\langle X \rangle$ and $\langle Y \rangle$ we get

$$\begin{aligned} ks + \sum_{i=1}^{t} (k-\lambda_i) + kt + \sum_{i=1}^{t} \lambda_i + sk & \\ &= 2k(s+t) \\ &= 2kp. \end{aligned}$$

Thus the number of edges in X and $Y = kp$. But, $\alpha_{1k}(G) + \beta_{1k}(G) \leq |X| + |Y| = kp$. □

Corollary 10a. *For a connected graph G with p vertices,*

$\alpha_{11}(G) + \beta_{11}(G) = \alpha_1(G) + \beta_1(G) = p$. *(Gallai)*

$\alpha_{11}(G) = \alpha_1(G)$, $\beta_{11}(G) = \beta_1(G)$ *and*

$\alpha_{11} + \beta_{11} = \alpha_1 + \beta_1 = 1 \cdot p = p$.

The following theorem is a natural extension of the Gallai Theorem II. The proof is essentially the same as that of Theorem 10.

Theorem 11. *Let f be an integer-valued function defined on $V(G)$ and suppose $f(v) \leqslant deg(v)$ for each $v \in V(G)$. Define $\alpha_{1f}(G)$ to be the minimum number of edges in an edge set $X \subseteq E(G)$ such that, for every $v \in V(G)$, $deg(v)$ in $\langle X \rangle$ is at least $f(v)$. Similarly $\beta_{1f}(G)$ is defined. Then*

$$\alpha_{1f}(G) + \beta_{1f}(G) = \sum_{v \in V(G)} f(v)$$

In [13], Laskar and Sherk showed that Gallai theorems can be found in the study of projective planes, as follows.
Let Π denote a projective plane of order n,
Let P denote the set of points of Π, and
Let L denote the set of lines of Π.
For any integer $m \geqslant 1$, let P_m denote a set of points of Π, such that for every $1 \in L$,

$$|1 \cap P_m| \geqslant m.$$

Similarly, let M_m denote a set of points of Π, such that for every $1 \in L$,

$$|1 \cap M_m| \leqslant m.$$

Finally, let α_m equal the minimum cardinality of all P_m-sets and let β_m equal the maximum cardinality of all M_m-sets.

Theorem 12 (Laskar, Sherk).

$$\alpha_m + \beta_{n+1-m} = n^2 + n + 1, \quad 1 \leqslant m \leqslant n.$$

We also note that a dual version of Theorem 11 also holds.

In [10] and [11], Hedetniemi and Laskar propose a bipartite theory of graphs, a theory in which many of the standard results in graph theory have counterparts for bipartite graphs. Among the results obtained in this bipartite theory are several 'bipartite' Gallai theorems, as follows.

Let $G = (X, Y, E)$ denote a bipartite graph where edges only join vertices in X with vertices in Y. Given an arbitrary graph $G = (V, E)$ we can construct three bipartite graphs of some interest, as illustrated in Fig. 2.

Let $G = (X, Y, E)$ be a bipartite graph.

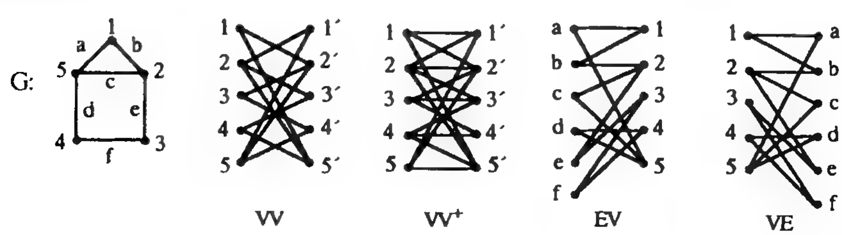

Fig. 2.

A set $S \subseteq X$ is a *Y-dominating set* if for every $y \in Y$ there is an $x \in S$ such that $(x, y) \in E$.
A set $S \subseteq X$ is *hyper-independent* if there does not exist a $y \in Y$ whose open neighborhood $N(y) \subseteq S$.
A set $S \subseteq X$ is *hyper-dominating* if for every $x \in X - S$ there does not exist a $y \in N(x)$ such that

$$N(y) \subseteq S \cup \{x\}.$$

Finally, define $\beta_h(G)$ to equal the maximum order of a hyper-independent set, and $\gamma_Y(G)$ to equal the minimum order of a Y-dominating set of G.

Theorem 13 (Hedetniemi, Laskar). *For any graph $G = (V, E)$, where $|V| = p$ and $|E| = q$,*

(i) $\gamma_t(G) + \beta_h(VV) = p$
(ii) $\gamma(G) + \beta_h(VV^+) = p$
(iii) $\alpha_1(G) + \beta_h(EV) = q$
(iv) $\alpha_0(G) + \beta_h(VE) = p$

A variety of new Gallai theorems can be generated from an extension of Theorem 2 which involves a generalization of the concepts of minimality and maximality. These ideas were introduced in [1]. We omit proofs here. We use the notation of Section 2.

A subset X of S is called a *k-minimal P-set* if X is a P-set but for all ℓ satisfying $1 \leq \ell \leq k$, all ℓ-subsets U of X and all $(\ell - 1)$-subsets R of S, $(X - U) \cup R$ is not a P-set. We note that 1-minimality is the usual concept of minimality.

Similarly X is a *k-maximal P-set* if X is a P-set but the addition of any ℓ elements to X where $\ell \leq k$ followed by the removal of any $\ell - 1$ elements, forms a set which is not a P-set. Let P be a hereditary property on S and let the new hereditary property Q on S be defined by:

X is a Q-set if and only if X is a transversal of the class of P'-sets.

Theorem 14 (Bollobás, Cockayne and Mynhardt). *X is a k-maximal P-set if and only if $S - X$ is a k-minimal Q-set.*

Let $\alpha_k(S, P)$ equal the minimum cardinality of a k-minimal P-set, and let $\beta_k(S, P)$ equal the maximal cardinality of k-maximal P-set.

Corollary 14a (Bollobás, Cockayne, Mynhardt). *Let P be a hereditary property on the subsets of a set S. Then*

$$\alpha_k(S, P) + \beta_k(S, P) = |S|.$$

Note that for $k = 1$, P-sets are independent sets of vertices and, again, we get

Gallai's theorem

$$\alpha_0 + \beta_0 = p.$$

As a final note, we mention that a matroid generalization of Gallai's theorem has been obtained by Kajitani and Ueno [12]. In particular, they establish a one to one correspondence between independent parity sets in a partition matroid M, induced by incidences of edges in the subdivision graph $S(G)$ of a graph G, and matchings in G.

Given this correspondence, it is easy to see that the complement of an independent parity set in the dual matroid M^* of M corresponds to a covering of G and vice versa. From this observation the following interesting result can be proved, a corollary of which is Gallai's Theorem II.

Theorem 15 (Kajitani and Ueno). *Let n be the number of vertices of a connected graph $G = (V, E)$.*

(i) *Suppose that X is a maximal matching of G. If Y is a covering of G which contains X and is minimal in the coverings containing X, then*

$$|X| + |Y| = n.$$

(ii) *Suppose that Y is a minimal covering of G. If X is a matching of G which is contained in Y and maximal in the matchings contained in Y, then*

$$|X| + |Y| = n.$$

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EDGE-PACKINGS OF GRAPHS AND NETWORK RELIABILITY

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The reliability of a network can be efficiently bounded using graph-theoretical techniques based on edge-packing. We examine the application of combinatorial theorems on edge-packing spanning trees, s, t-paths, and s, t-cuts to the determination of reliability bounds. The application of spanning trees has been studied by Polesskii, and the application of s, t-paths has been studied by Brecht and Colbourn. The use of edge-packings of s, t-cutsets has not been previously examined. We compare the resulting bounds with known bounds produced by different techniques, and establish that the edge-packing bounds often produce a substantial improvement. We also establish that three other edge-packing problems arising in reliability bounding are NP-complete, namely edge-packing by network cutsets, Steiner trees, and Steiner cutsets.

1. Edge-packings of graphs

An *edge-packing* of a multigraph G is a collection of edge-disjoint subgraphs of G. Many combinatorial problems can be viewed as edge-packing, including the chromatic index problem and the independent set problem. In this paper, we are concerned with edge-packing problems which arise in problems concerning network reliablity. The three problems of most interest to us here are edge-packings by spanning trees, by s, t-paths, and by s, t-cuts. We first review the use of edge-packings in the reliability context.

A *probabilistic graph* is a multigraph together with a probability of operation for each edge. Vertices of the multigraph represent communication centers which never fail, while edges represent undirected communication links which operate statistically independently with the stated probabilities. An *all-terminal* operation in such a graph requires that all vertices be able to communicate with one another via paths of operational edges, and *all-terminal reliability* is the probability that the network supports an all-terminal operation. A *2-terminal* operation for specified vertices s and t requires that there be an operational s, t-path, and *2-terminal reliability* is the probability that a 2-terminal operation for s and t can be performed. A *k-terminal* operation for a specified set of k *target nodes* requires that each pair of target nodes be able to communicate, and *k-terminal reliability* is the probability of being able to carry out a k-terminal opeation. The evaluation of

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each of these three reliability measures is a very difficult problem; in fact, they are $\#P$-complete [19, 23]. This had led network designers to consider lower and upper bounds on the reliability which can be efficiently computed. The goal of this paper is to study the application of edge-packings to the production of reliability bounds. We also compare some computational results for the bounds with bounds obtained from the other main technique, subgraph counting (see [6]). It is also important to remark on the existence of efficient methods for combining the results of the edge-packing bounds described here with the subgraph counting bounds; see [5, 7].

We illustrate the general method for obtaining lower bounds by showing how an edge-packing by spanning trees leads to a lower bound on all-terminal reliability. Let G be a probabilistic graph whose all-terminal reliability is desired, and let $\{T_1, \ldots, T_k\}$ be an edge-packing of G by spanning trees. Observe that the all-terminal reliability R_i of the tree T_i is easily computed; it is just the product of the edge operation probabilities of the edges in the tree. Now every state of the edges of G corresponds to a state of the edges in each tree; a failed state of G must yield failed states in each tree, but the converse need not hold. Thus an upper bound on the failure probability of G is the probability that all of the trees fail; taking complements of each gives a lower bound on the probability that G operates, as desired. The bound will be improved in general by having more trees, and by having more reliable trees (these two goals may be conflicting). The simplest edge-packing problem of interest here is to pack the maximum number of spanning trees. The application to 2-terminal lower bounds is very similar, except that we consider edge-packings by s, t-paths. In the application to k-terminal reliability, we consider edge-packing by *Steiner trees*, subtrees connecting all target nodes in which each leaf is a target node.

Upper bounds arise in a very similar way. A cutset is any set of edges whose removal prevents network operation. Cutsets whose removal disconnects s from t are *s, t-cutsets,* and those which disconnect the network (i.e. prevent all-terminal operation) are called *network cutsets.* Finally, cutsets which separate any two target nodes are *Steiner cutsets.* To develop an upper bound, consider the use of edge-packing by cutsets to bound reliability. Let G be a graph, and $\{C_1, \ldots, C_k\}$ be an edge-packing by cutsets for G. An operational state of G corresponds to a state of each cutset in which each has at least one operational edge, although the converse need not hold. Hence the probability that each cutset has at least one operational edge is an upper bound on the reliability of G. This can be easily evaluated, as the cutsets are edge-disjoint. The bound is improved by taking more cutsets, and cutsets which are more likely to fail; again, these two goals may be conflicting. We consider the simplest case of requiring an edge-packing by a maximum number of cutsets.

Two main points are at issue here. First, we must consider whether the bounds developed thus far can be efficiently computed. Second, we must ascertain

whether the resulting bounds are of any practical use—in particular, whether they are competitive with the subgraph counting bounds. Edge-packing by spanning trees, s, t-paths, and s, t-cuts all have efficient solutions, while edge-packing by network cutsets is NP-complete (see Section 6). In the last case, however, the lack of an efficient edge-packing algorithm is more than compensated for by a clever bounding technique due to Lomonosov and Polesskii [14] which employs 'noncrossing' rather than edge-disjoint cutsets; noncrossing cuts can be found efficiently using the Gomory-Hu algorithm. Edge-packing by Steiner trees and Steiner cutsets are both NP-complete; we establish this in Section 5 and 6. We first focus on the three efficiently solvable edge-packing problems. The application of the first two in bounding reliability has been studied previously; we review these, and then examine more carefully upper bounds on 2-terminal reliability from edge-packings by s, t-cutsets.

2. Edge-packing spanning trees

Tutte [22] and Nash-Williams [17] first addressed the problem of edge-packing a graph with spanning trees; Edmonds [8] subsequently obtained a more powerful result which gives conditions under which a matroid contains k disjoint bases. The application to edge-packing spanning trees is immediate, as the spanning trees are precisely the bases of the graphic matroid of the graph. Polesskii [18] observed that the Tutte–Nash–Williams theorem has a particularly useful corollary: a graph with edge-connectivity $2t$ or $2t+1$ has at least t edge-disjoint spanning trees. This corollary leads to a simple expression for a lower bound when each edge has operation probability p: an n-vertex graph with edge-connectivity $2t$ or $2t+1$ has all-terminal reliability at least $1-(1-p^{n-1})^t$. This is called the *Polesskii bound*, and is of interest largely because it requires no knowledge of the actual spanning trees, or a method to find them—it requires only the number of vertices and the edge-connectivity.

One way to improve the Polesskii bound is to employ the actual number of edge-disjoint spanning trees. Fortunately, Edmonds's proof of the sufficiency of the Tutte–Nash–Williams conditions is based on an elegant efficient algorithm, the *matroid partition* algorithm. The matroid partition algorithm can be used to construct efficiently a maximum set of edge-disjoint spanning trees. In addition to incorporating the actual number of trees, knowledge of the actual trees enables us to allow different edge probabilities.

It is difficult to compare the resulting edge-packing bound with other available bounds, to ascertain whether it affords some practical improvement. Two main competitors exist: the *Ball–Provan bounds* [2] are the best subgraph counting bounds which apply, and the *Lomonosov–Polesskii lower bound* [15] also applies; however, both assume that all edge-probabilities are equal, while the

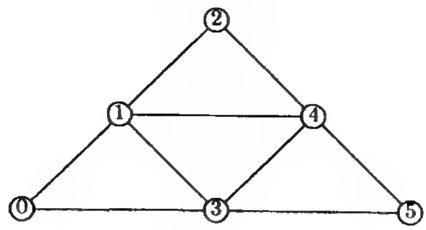

Fig. 1.

edge-packing bound requires no such assumption. In fact, the edge-packing bound is the only basic bound which applies here when different edge probabilities are allowed, and hence in this context is very valuable indeed!

When all edge operation probabilities are equal, it is of interest to compare the edge-packing bound to the Ball–Provan and Lomonosov–Polesskii bounds. Unfortunately, in our experience, the edge-packing bound fares quite poorly. Let us consider a small example on six vertices, with each edge having operation probability p, depicted in Fig. 1.

The three bounds evaluated on this graph yield:

Edge-packing	p^5
Lomonosov–Polesskii	$6p^5 - 5p^6$
Ball–Provan	$p^5[1 + 4q + 7q^2 + 10q^3 + 14q^4 + 18q^5]$ $(q = 1 - p)$.

For this example, both bounds always beat the edge-packing bound. This unfortunate state of affairs persists in other examples. This appears to be a serious setback for the edge-packing approach, and may account for the lack of attention afforded it. However, when we turn to the 2-terminal problems, edge-packing is a winner.

3. Edge-packing s, t-paths

Menger's theorem [16] guarantees that the number of edge-disjoint s, t-paths equals the minimum cardinality of an s, t-cutset, and hence network flow algorithms give an efficient method for edge-packing a graph with the maximum number of s, t-paths. Brecht and Colbourn [4] examined the application of edge-packings of s, t-paths to reliability bounds, and found that the edge-packing bounds are not only competitive with, but often much better than, other available bounds. Once again, no other available bound allows different edge probabilities. When all edge probabilities are equal, the best subgraph counting bound which applies is the Kruskal-Katona bound. Even in this case, the edge-packing bound is better typically in our test cases. A small example to illustrate this is to take the

graph of Fig. 1 with $s = 0$, $t = 5$. For various values of p, we find the following:

p	edge-packing	Kruskal-Katona	exact
0.2	0.0476800	0.0400041	0.0609183
0.3	0.114570	0.0900919	0.154691
0.4	0.213760	0.160786	0.293976
0.5	0.343750	0.253906	0.464844
0.6	0.498240	0.373437	0.642341
0.7	0.664930	0.524589	0.798801
0.8	0.824320	0.707109	0.913646
0.9	0.948510	0.896093	0.979453

For this example, we chose an edge-packing with a path of length two and a path of length three. There are other edge-packings; in particular, for this example, one might obtain a path of length four and a path of length five. This selection would lead to much poorer bounds, despite the fact that it remains an edge-packing with a maximum number of paths.

One of the main open questions here is which selection of paths leads to the best bound. It is easy to see that, when all other things are equal, more paths are better. Similarly, if the paths are more reliable (shorter in the equal probability case), the bound is better. For this reason, Brecht and Colbourn [4] modified the edge-packing bound in two ways. First of all, notice that it is not always best to take the maximum number of paths; given a choice between one short path or two long paths, the better selection will depend on the value of p. One can exploit the structure of network flow algorithms to help out here; flow algorithms produce edge-packings by k paths from edge-packings by $k-1$ paths, and hence one can retain the best bound resulting from all number of paths from 1 up to the s, t-edge-connectivity. Another improvement can be made by attempting to reduce the average length of paths used; this can be accomplished by using a *minimum cost* network flows algorithm, to produce a set of paths whose total length is minimum. Both of these heuristics have been found to be quite useful in tightening the bounds [4].

Edge-packing is largely vindicated as a bounding strategy because of the results for 2-terminal lower bounds. It is interesting to note that the availability of a powerful minmax theorem, Menger's theorem, is key to the success of the approach. One of the important areas for study here is the determination of the *best* edge-packing bound; even the mincost strategy is just a heuristic method.

4. Edge-packing s, t-cutsets

Edge-packing s, t-cutsets has been examined by Robacker [21], and subsequently by Fulkerson [9, 10] as a dual of the maxflow-mincut theorem. A

characterization of the maximum number of edge-disjoint s, t-cuts, and an algorithm to find them, is remarkably simple.

Theorem. *The maximum number of edge-disjoint s, t-cutsets equals the minimum length of an s, t-path.*

Proof. An s, t-cut contains at least one edge from every s, t-path. Thus, if there is an s, t-path of length l, there can be at most l edge-disjoint s, t-cutsets. In the other direction, if l is the length of a shortest s, t-path, we can efficiently construct l edge-disjoint s, t-cutsets as follows. Starting at vertex s, label the vertices (using a breadth-first search) with their distance from s. An s, t-cutset is obtained by taking all edges from vertices at distance $i-1$ to vertices at distance i. A collection of l cutsets is obtained by repeating this process for each i from 1 to l; the l s, t-cutsets which result are edge-disjoint. This completes the proof. □

As in the case of s, t-paths, this easy theorem gives a strong minmax theorem for edge-packing s, t-cutsets. One might therefore reasonably expect the resulting edge-packing bound to be powerful. Once again, the only serious competitor here is the Kruskal-Katona bound [24]. To effect a comparison, we assume equal edge operation probabilities; the Kruskal-Katona bound requires this. Rather than content ourselves with small contrived examples, we compare the two bounds on the 1979 Arpanet, a real computer network whose analysis is of practical interest [3]; see Fig. 2. This network has 59 vertices and 71 edges. We should remark in advance that the s, t-cutsets produced by simple breadth-first search are not necessarily minimal; we always modify the edge-packing produced to make the cutsets minimal.

We first consider two nodes which are 'far apart', nodes 20 and 41. These two nodes are at distance ten, the diameter of the Arpanet. The theorem guarantees that ten edge-disjoint cutsets can be found. Labelling the vertices via breadth-first search and making each cutset minimal gives the following ten cutsets:

$\{15,20\}, \{19,20\}$

$\{8,15\}, \{17,19\}$

$\{5,8\}, \{17,18\}$

$\{1,5\}, \{5,22\}, \{16,18\}, \{18,18\}$

$\{22,24\}, \{1,2\}, \{14,16\}, \{27,28\}, \{28,55\}$

$\{24,31\}, \{2,3\}, \{2,4\}, \{13,14\}, \{27,30\}, \{46,55\}, \{53,55\}, \{55,57\}$

$\{31,33\}, \{31,42\}, \{29,30\}, \{43,46\}, \{50,53\}, \{57,58\}$

$\{33,38\}, \{39,42\}, \{42,45\}, \{29,32\}, \{43,49\}, \{40,50\}, \{50,52\}, \{58,59\}$

$\{38,48\}, \{39,56\}, \{37,39\}, \{45,47\}, \{32,34\}, \{44,49\}, \{49,51\}, \{35,40\}, \{52,54\}, \{56,59\}$

$\{37,41\}, \{41,44\}$

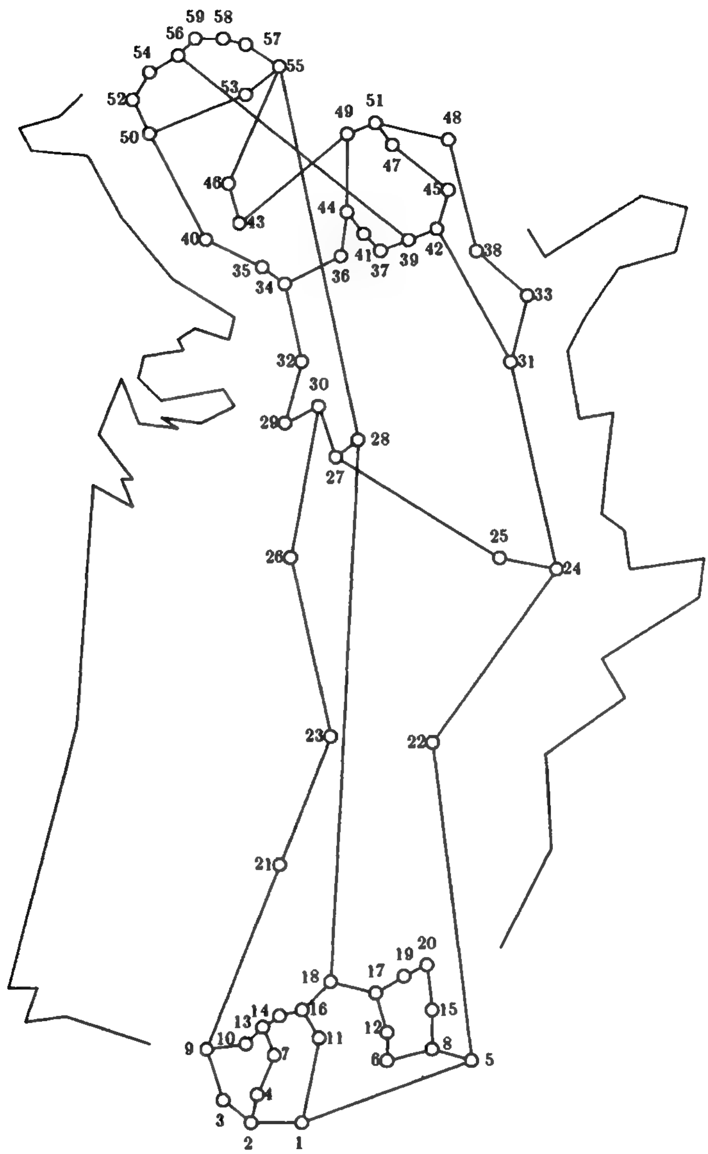

Fig. 2.

Of course, this collection of cutsets is not unique, nor do we have any assurance that it leads to the best bound. Nevertheless, we can employ any edge-packing to obtain a bound, and this it is of interest to see whether this simple strategy for producing an edge-packing leads to a competitive bound. We therefore compare the upper bound arising from this edge-packing with the Kruskal-Katona upper bound. The resulting bounds for various values of p are

summarized below:

p	Kruskal-Katona	edge-packing
0.1	0.029209	0.000018
0.2	0.328144	0.003041
0.3	0.509623	0.032565
0.4	0.640000	0.123363
0.5	0.750000	0.280391
0.6	0.840000	0.477515
0.7	0.910000	0.677955
0.8	0.960000	0.847658
0.85	0.977500	0.912450
0.9	0.990000	0.960489
0.92	0.993600	0.974601
0.94	0.996400	0.985664
0.96	0.998400	0.993613
0.98	0.999600	0.998401

The edge-packing bound is dramatically better than the Kruskal-Katona bound for this example. Of course, the Kruskal-Katona bound is hampered in this case by the fact that the edge-connectivity is only two; in addition, the edge-packing bound is assisted by the fact that the nodes chosen are at great distance. For this reason, it is worthwhile to examine another example; to obtain favourable conditions for the Kruskal-Katona bound, we select nodes which are highly connected. To obtain unfavourable conditions for the edge-packing bounds, we select nodes which are close together. Consider taking $s = 24$, $t = 31$. Only one cutset can be included, and it has size 3 at least. Naturally, we expect a much poorer bound as a result (although one must be careful, as the reliability of communication between these two nodes is actually much higher than the previous pair used). This second example is of interest primarily because it seems to be the case where the Kruskal-Katona bound is the best in comparison to the edge-packing bound.

Remarkably, while the Kruskal-Katona bound does 'win' for small values of p ($p < 0.18$), past this point the two bounds *agree* to six digits of accuracy. This is quite remarkable, as one might expect the easily evaluated edge-packing bound to be very poor compared to the more sophisticated Kruskal-Katona bound. In a sense, the results here reflect on the poor quality of the 2-terminal upper bounds from the Kruskal-Katona method. Nevertheless, they do suggest that edge-packing is not only a viable competitor, but is a practical improvement. Our purpose here is to illustrate that 2-terminal upper bounds from edge-packing are just as promising as the useful 2-terminal edge-packing lower bounds. However, much research remains to be done in this area. Primarily, one wants a characterization of which edge-packings by s, t-cutsets lead to the best bound, and efficient algorithms to find such edge-packings.

5. Edge-packing Steiner trees

Turning to k-terminal lower bounds, the situation is not as satisfactory; we show here that edge-packing by Steiner trees is NP-complete. The Steiner tree problem is a well-known NP-complete problem [11], in which one is required to produce a Steiner tree with a bound on the number (or total weight) of edges used. Edge-packing Steiner trees carries no such stipulation, and hence its NP-completeness does not follow trivially from the NP-completeness of the Steiner tree problem. Nevertheless, we are able to prove NP-completeness via a transformation the NP-complete problem of determining the chromatic index of a graph.

Theorem. *Let G be a graph, and T a set of target vertices of G. Determining whether G has an edge-packing of at least k Steiner trees for T is* NP*-complete for any fixed $k > 2$.*

Proof. Membership in NP is straightforward; one need only guess k edge-disjoint subgraphs, and verify that each is a Steiner tree for T. Completeness is shown using a reduction from chromatic index of k-regular graphs, which is NP-complete for every fixed $k > 2$ [12, 13].

Let $G = (V, E)$ be a k-regular graph, an instance of the chromatic index problem. Let $V = \{v_1, \ldots, v_n\}$, $E = \{e_1, \ldots, e_m\}$, and $F = \{f_1, \ldots, f_m\}$. Form a graph H as follows. The vertices of H are the elements of V, E, and F together with $\{c_1, \ldots, c_k\}$ and a distinguished element *root*. The edges of H are as follows:

1. for each $e = \{v, w\}$ in E, $\{e, v\}$ and $\{e, w\}$ are edges.
2. for each i, $\{e_i, f_i\}$ is an edge.
3. for each e_i, c_j, $\{e_i, c_j\}$ is an edge.
4. for each c_j, $\{root, c_j\}$ is an edge.

Finally, the target set T consists of V together with *root*. We claim that H has an edge-packing by k Steiner trees for T if and only if G is k-edge-colourable.

Suppose G has chromatic index k. Each colour class in the edge-colouring forms a 1-factor of G; suppose that colour i contains the edges with indices I. Then form a Steiner tree S_i by taking all edges induced on the set of vertices consisting of *root*, c_i, all elements in E and F with indices from I, and all elements of V. The set of vertices so defined induces a Steiner tree on the target vertices, since the edges in a colour class share no vertices. Moreover, since every two colour classes are edge-disjoint, so are the Steiner trees produced.

In the other direction, suppose that H has an edge-packing of k Steiner trees for T. Each element of V has degree k in H, and appears in each Steiner tree. Hence, each has a unique neighbour in each Steiner tree. In total, there are $\frac{1}{2}kn$ edges from elements of E to elements of F, which we call *cross* edges. Now since each element of V has a unique neighbour in E, at least $\frac{1}{2}n$ cross edges appear in

a Steiner tree. Since there are k edge-disjoint Steiner trees, each must therefore contain precisely $\frac{1}{2}n$ cross edges; in fact, the Steiner trees partition the cross edges. Consider a class in this partition. It consists in G of a set of edges which meets every vertex—that is, a 1-factor. But then the Steiner trees give a partition of G into k 1-factors, which is just a k-edge-colouring. This completes the proof. □

The NP-completeness here severely limits the use of edge-packing bounds for reliability in this case. One could, of course, settle for edge-packings of close to maximum size; no research along these lines seems to have been undertaken.

6. Edge-packing network cutsets

In this section, we consider edge-packing the maximum number of network cutsets. Since this is the special case of edge-packing Steiner cutsets in which all vertices are targets, an NP-completeness result here establishes NP-completeness for edge-packing Steiner cutsets as well. It is instructive to first observe two lower bounds on the maximum size of an edge-packing. Every s, t-cutset is a network cutset, and hence an edge-packing by s, t-cutsets is also an edge-packing by network cutsets. This holds for any choice of s and t, and hence a lower bound on the maximum size of an edge-packing is the diameter of the graph.

Another lower bound results from considering independent sets of a graph. For each vertex of a graph, define the *star* of the vertex to be the set of edges incident with the vertex. The star of a vertex is a network cutset. Taking stars of all vertices in an independent set gives an edge-packing by network cutsets, and hence the maximum size of an edge-packing is at least the maximum size of an independent set. Since the independent set is NP-complete, the NP-completeness of edge-packing network cutsets would be trivial if these numbers always agreed; however, there are graphs whose diameter exceeds the size of their maximum independent set (paths, for example) and these graphs have maximum edge-packings whose size exceeds the size of their maximum independent set. Nevertheless, we can use a simple transformation to reduce independent set to edge-packing network cutsets:

Theorem. *Determining whether a graph G has an edge-packing by at least k network cutsets is NP-complete.*

Proof. One can guess a set of k edge-disjoint subgraphs, and verify that each is a network cutset, and all are edge-disjoint; hence the problem is in NP. Completeness is shown by a reduction from determining whether a graph has an independent set of size at least k, a well-known NP-complete problem.

Let G, k be an instance of the independent set problem. Transform G to

construct a new graph H; H is a copy of G, with a new vertex *dom* added which is adjacent to all of the other vertices. We claim that G has an independent set of size at least k if and only if H has an edge-packing by at least k network cutsets. First notice that the size of a maximum independent set in G is the same as that in H.

Now suppose that H has an independent set of size k. Taking the stars of these k vertices gives an edge-packing of H by k network cutsets.

In the other direction, let H have an edge-packing by k network cutsets. Without loss of generality, each cutset is minimal. Hence, each cutset partitions the vertex set of H into two connected pieces. For the ith cutset, let X_i denote the vertices in the resulting connected piece which does not contain *dom*. Since the network cutsets are edge-disjoint, no two of the $\{X_i\}$ share a vertex. In fact, no vertex in X_i has a neighbour in any X_j unless $i=j$. But then selecting a vertex from each X_i gives an independent set of size k, as required. This completes the proof. □

Once again, the NP-completeness here limits the applicability of the edge-packing bounds. In this case, however, two methods have been proposed for avoiding this complexity. One method is to employ 'noncrossing' cutsets, rather than edge-disjoint ones; here, an efficient algorithm, the Gomory–Hu algorithm, does exist. The application to reliability, due to Lomonosov and Polesskii [14], gives useful bounds. A second method is to transform the undirected problem to a directed one, and to use arc-disjoint directed cutsets; see [20].

7. Concluding remarks

The main conclusion is that edge-packing provides a viable strategy for producing reliability bounds which are competitive with the best subgraph counting bounds. In the all-terminal case, the presence of a very powerful subgraph counting method combines with the lack of "dense" edge-packings to make the edge-packing bound relatively poor. However, in 2-terminal lower and upper bounds, a weaker subgraph counting method combined with very powerful minmax results on edge-packing make the edge-packing bounds typically better. Improvements in the subgraph counting methods seem unlikely to compensate for the big improvement which the edge-packing bounds sometimes afford. Nevertheless, an analogue of the Ball–Provan bounds for the 2-terminal problems would be very useful.

In conclusion, we should also remark that the edge-packing and subgraph counting methods both apply to reliability problems on directed graphs as well. In fact, any undirected reliability problem can be easily transformed to a directed one [1]. One might therefore examine transforming an undirected problem to a directed one and then employing arc-packing bounds on the result. In the

2-terminal problems, the bounds are identical. However, in the all-terminal problem, a substantial improvment in the all-terminal bounds is obtained; this is explored further in [20].

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TWO HAMILTON CYCLES IN BIPARTITE REFLECTIVE KNESER GRAPHS

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Let i and j be distinct positive integers. Let $RG_{i,j}$ be the bipartite graph whose vertices are the i- and j-subsets of $\{0, 1, \ldots, i+j-1\}$ and whose adjacency is given by inclusion. We consider an Erdös Conjecture that asserts that the graphs $RG_{i,j}$ are hamiltonian. In this paper Hamilton cycles in $RG_{9,10}$ and $RG_{7,9}$ are constructed by means of techniques involving group actions on graphs and their quotient graphs.

1. Introduction and background

Let $i, j \in Z$ with $0<i<j$ and let $n=i+j$. Let $V_{i,j}$ be the family of $(i+j)$-tuples over $Z_2=\{0, 1\}$ whose Hamming weight is either i or j. Let $RG_{i,j}$ be the bipartite graph whose set of vertices is $V_{i,j}$ with adjacency given by changing exactly $j-i$ coordinates of value 0 to value 1 or viceversa. For $j=i+1$ we have treated in [1] and [4] the hamiltonicity problem on $RG_{i,i+1}$, which is an Erdös Conjecture on subsets of $\{0 \cdots i+j-1\}$ in its vectorial version.

If Γ is a group then a group action $\tau:\Gamma\times G\to G$ on a graph $G=(V, E)$ is defined in [2] as a pair $(T:\Gamma\times V\to V, t:\Gamma\times E\to E)$ of compatible actions. In fact $RG=RG_{i,j}$ admits some group actions that ease the study of their hamiltonicity. To describe them it is enough to indicate their vertex set action components:

(0) *Back-reading of coordinates*:

$$T_0: Z_2\times RG\to RG,$$

where $T_0(1, a)=(a_{i+j-1}, \ldots, a_0)$, for $a=(a_0, \ldots, a_{i+j-1})$ and $1\in Z_2$.

(1) *Complementation*:

$$T_1: Z_2\times RG\to RG,$$

where $T_1(1, a)=(1+a_0, \ldots, 1+a_{i+j-1})$, where a and 1 are as before.

(2) *Rotation of coordinates*:

$$T_2 : Z_{i+j} \times \mathrm{RG} \to \mathrm{RG},$$

where $T_2(\mu, a) = (a_{-\mu}, a_{-\mu+1}, \ldots, a_{-\mu-1})$, for every $\mu \in Z_{i+j}$ and a as above.

(3) *r-interleaving of coordinates, where* $1 < r \in Z$ *and g.c.d.* $(r, i+j) = 1$:

$$T_3 : Z_{\Phi(i+j)} \times \mathrm{RG} \to \mathrm{RG},$$

where $\Phi(i+j)$ is the Euler totient and may be replaced by the order k of $r \bmod i+j$, i.e. the lowest positive integer k such that $r^k \equiv 1 \pmod{i+j}$. For example, $T_3(-1, a) = (a_0, a_r, \ldots, a_{-r})$.

Let $T_{01} : Z_2 \times \mathrm{RG} \to \mathrm{RG}$ be the action obtained by composition of T_0 and T_1, that is back-reading plus complementation. In [1] we showed that there exists a quotient graph $\mathrm{RH}_{i,i+1}$ of $\mathrm{RG}_{i,i+1}$ such that any Hamilton path bearing some mild conditions can always be lifted to a Hamilton cycle in $\mathrm{RG}_{i,i+1}$. The mild conditions are:

(a) Each (resp. one) end of the Hamilton path in $\mathrm{RH}_{i,i+1}$ has at least one loop (resp. two loops).

(b) If $n = 2k + 1$ is composite, the Hamilton path passes through a 2-link.

We will call such a path a cyclable path.

In fact we define $\mathrm{RH}_{i,j}$ as the quotient graph of $\mathrm{RG}_{i,j}$ under the actions τ_{01} and τ_2.

Example. Represent vertices in $\mathrm{RG}_{i,j}$ as $(i+j)$-words in the alphabet $\{0, 1\}$. If a is such a word let (a) be its orbit in $\mathrm{RG}_{i,j}/\tau_2$, that is (a) is the circular vector given by a. Let us represent elements in $\mathrm{RH}_{i,j} = \mathrm{RG}_{i,j}/\tau_{0,1}, \tau_2$ by circular vectors of Hamming weight j. Thus for $i = 2$ and $j = 3$ we have $\mathrm{RH}_{2,3}$ as represented in Fig. 1(a) where each vertex has two additional loops. The induced adjacency in $\mathrm{RG}_{2,3}$ goes as follows: Consider for example the vertex (11100). Select a coordinate with value 1. There is an adjacent vertex to (11100) that reads backward as follows:

(a) Write down again 1 for the chosen coordinate.

(b) Write down all the other values changed, for the remaining coordinates.

(c) Do not forget that the result has to be read backwards, or better be written down backwards.

If this induced adjacency is modified by forgetting the back-reading, then we get the graph $H_{2,3}$ of circular vectors of the Kneser graph $K_{2,3}$. Its lifting $G_{2,3}$

Fig. 1

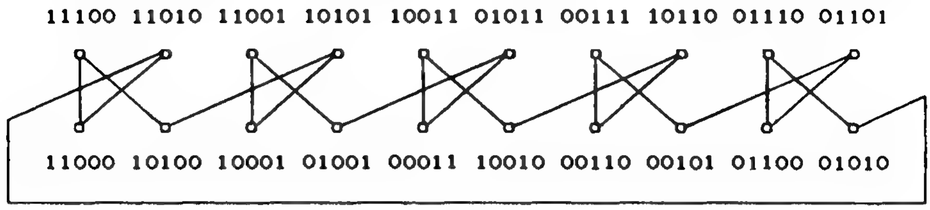

Fig. 2.

amounts to changing the adjacency of $RG_{2,3}$ by means of the back-reading of adjacent vectors.

$RG_{2,3}/\tau_2$ is a lifting of $RH_{2,3}$, represented in Fig. 1(b), where the 2-links are to be noticed. For example the 2-link on the left comes from the fact that 11100 is adjacent to both 11000 and 01100. A cyclable path in $RH_{2,3}$ is obtained easily and lifts $RG_{2,3}/\tau_2$ to the Hamilton cycle in $RG_{2,3}$ represented in Fig. 2. Note that a pattern is repeated five times by only rotating uniformly the coordinates one position to the left each time, where the position previous to the 0th coordinate is the $(i+j)$th one. Also note that the vertices in the Hamilton cycle above were arranged in such a way that the pair of vertices in each column is an orbit of T_{01}. This disposition makes the whole graph 11100 symmetric with respect to an horizontal middle line. The same property holds for any $RG_{i,j}$, see [2] Theorem 2.

In [3] the method described in the example was applied and Hamilton cycles were found this way for $i = j - 1 \leq 8$, and for odd $i = j - 2 \leq 6$. Variations of this method produced in [3] Hamilton cycles in $RG_{i,j}$, for even $i = j - 2 \leq 6$ and for $i = j - 3 = 3$.

In [1] a characterization of 2-looped vertices and 2-links in $RH_{i,i+1}$ was given. For example $(11 \cdots 10 \cdots 0)$ is a 2-looped vertex in $RH_{i,i+1}$. Also the other elements in the orbit of $(11 \cdots 10 \cdots 0)$ by the r-interspersion action T_3 are 2-looped vertices. Note that they are also back-reading symmetrical or palindromic, that is invariant under T_0. In general, $(11 \cdots 10 \cdots 0) \in RG_{i,j}$ will be a $(j-i+1)$-looped vertex. We choose these looped vertices generally as the ends of our cyclable paths.

In Section 2 a cyclable path in $RH_{9,10}$ found by Cordova for his Master Thesis is given. A convenient variation of the described group action approach allowed Quintana to find a Hamilton cycle in $RG_{7,9}$ for his Master's work and this is given in Section 3.

2. A cyclable path in $RG_{9,10}$

If τ is a Γ-action on G, then there is a quotient graph $G/\tau = G/\Gamma = (V/T, E/t)$ which is a *labelled directed quotient graph* as follows:

(1) Each element A of the V/T is labelled with the cardinality $|A|$ of it as an equivalence class in V.
(2) In each $A \in V/T$ select a *main representative* $A.0 \in A$ of A.
(3) Transform each edge $\bar{e} \in E/t$ into an arc $e^\wedge$ with arbitrary orientation. If the arc $e^\wedge$ of an edge $\bar{e} \in E/t$ goes from A to B, then label $\bar{e}$ with the integer $z \in [0, \gcd\{|A|, |B|\}]$ such that the unique representative $e \in E$ of $\bar{e}$ with an end $A.0$ has $B.z = T(z.1, B.0)$ as its other end, where 1 is the unit generator of Γ.

A path in $RG_{i,j}/\tau_2$, made up by two zigzag crossed nonhorizontal paths joined by a horizontal link, or any of its liftings, in $RG_{i,j}$, is called a *spring*.

The vertices of $RG_{i,j}/T_2$ (resp. $RH_{i,j}$) are the circular vectors arising from the vectors in $C_i \cup C_j$, (resp. C_j). A circular vector will be given as a word in parentheses, so that if $(A:0 = (a_0a_1 \cdots a_{n-1}),\ A:1 = (a_1a_2 \cdots a_0), \ldots, A:(n-1) = (a_{n-1}a_0 \cdots a_{n-2})$, then $(A:0) = (A:1) = \cdots = (A:(n-1))$.

To determine Hamilton cycles in $RG_{i,j}$ we will consider the quotient graph $RJ_{i,j} = RH_{i,j}/T_3$. We denote any vertex $A = [a] \in RJ_{i,j}$, for $a \in RG_{i,j}$, as to have main representative $A.0 = (a) \in RH_{i,j}$. We are interested in the T_3-orbit cardinality (oc) of each element of $RJ_{i,j}$. In (the subset of palindromic vertices of) $RJ_{i,i+1}$, there is a maximal oc, denoted (pmoc(n)), resp. moc(n). According to whether τ_0 is a restriction of τ_3 or not, $\mathrm{pmoc}(n) = \frac{1}{2}\mathrm{moc}(n)$ or $\mathrm{pmoc}(n) = \mathrm{moc}(n)$, for every positive $i \in Z$. In [3] we found it convenient to try to find for $i > 4$ a subgraph of $RJ_{i,i+1}$ consisting of: (a) Either a collection $\{P\}$ of disjoint paths with the ends, (resp. inner vertices), of each P having oc = pmoc(n), (resp. moc(n)); (b) Or a collection $\{S\}$ of disjoint cycles on vertices with oc = moc(n). Each P as in (a) can be blown up to a cycle $S = S(P)$. By breaking up the $S(P)$'s into paths and plugging to their ends the remaining vertices, we constructed cyclable paths, for $i = 5, 7, 8$. For $i = 6$, a similar method was utilized out of (b). For all of this, if $S = (\ldots, A.s, B.s, \ldots)$ is oriented, then we denote $^\wedge A.s = (B.s)^{-1})$ for $S - (A.s, B.s)$ as an oriented path from A to B. The path obtained by deleting (resp. considering only) from $A^\wedge .s$, its last y vertices is denoted by $A^\wedge ;y.s$, (resp. $A^\wedge y;.s$). This is the ';' convention which will be used in this section and in the following one.

A path p in $RH_{i,j}$ is given by its first vertex $\Phi(P) = P_1$ and the sequence $\pi(P) = \{\pi_s(P)\}$ of the Γ-coordinates (tuples in Section 3), that remain unchanged from the sth vertex of P to the $(s+1)$th one.

A collection of paths in $RJ_{9,10}$ as in (a) above is constituted by the following paths $P, Q, \ldots, Z$ where the end vertices have oc = 9 and the inner vertices have oc = 18:

$\Phi(P) = [1111111111000000000]$, $\pi(P) = \{1, 18, 9, 17, 8, 16, 7, 14, 6, 13, 14, 6, 16, 8, 2, 1, 6, 2, 1, 12, 2, 6, 13, 10, 4, 7, 17, 9, 5, 4, 6, 17, 10, 2, 9, 14, 0, 16, 3, 10, 17, 1, 2, 0, 4, 3, 8, 6, 10, 5, 16, 11, 14, 7, 1, 4, 0\}$;

$\Phi(Q) = [10111111111010000000]$, $\pi(Q) = \{5, 18, 0, 10, 11, 0, 10, 17, 18, 12, 17, 10, 2, 13, 4, 15, 3, 5, 8, 11, 5, 17, 11, 8, 0, 5, 7, 11, 9, 7, 5, 16\}$;

$\Phi(R) = [01111111100100000010]$, $\pi(R) = \{4, 0, 3, 12, 17, 16, 2, 9, 7, 3, 12, 13, 1, 12, 5, 10, 6, 15, 10, 14, 9, 10, 11, 17, 10, 11, 12\}$;

$\Phi(S) = [01111111100000101000]$, $\pi(S) = \{3, 10, 4, 0, 6, 9, 1, 17, 2, 6, 10, 3, 7, 12, 5, 2, 0, 7, 13, 0, 9, 4, 0, 13, 17, 11, 6, 1, 4, 5, 1, 6, 5, 18, 8, 1, 6\}$;

$\Phi(T) = [11110111111000000001]$, $\pi(T) = \{2, 17, 3, 14, 0, 4, 6, 15, 4, 12, 18, 13, 14, 2, 9, 0, 12, 6, 15, 3, 8, 9, 1, 10, 6, 18, 7, 4, 3, 7, 5, 8, 7, 5, 10, 12, 0, 14, 4\}$;

$\Phi(U) = [01111111100001000100]$, $\pi(U) = \{7, 0, 5, 13, 0, 5, 16, 7, 1, 0, 6, 14, 4, 6, 0, 17, 13, 9, 15, 3, 11, 5, 13, 11, 1, 9, 4, 13, 3, 6, 14, 7, 10, 15, 9, 14, 5, 9, 13, 3, 18, 13, 6, 8, 15, 16, 4, 11, 1, 0, 12, 15, 18, 12, 17, 4, 5\}$;

$\Phi(V) = [10111111010000101000]$, $\pi(V) = \{6, 1, 5, 12, 13, 16, 7, 14, 15, 8, 2, 10, 16, 13, 14, 5\}$.

The other vertices of $\mathrm{RJ}_{9,10}$ are:

$$\begin{aligned} \mathrm{oc} = 6\colon\ & W = [10110111111000010010];\\ & X = [11010111111001000010];\\ & Y = [10111101101001000010];\\ \mathrm{oc} = 2\colon\ & \Sigma = [10110011110101000001]. \end{aligned}$$

The following paths exist for $0 \leqslant s \leqslant 9$:

$\sigma.s = (X.s,\ Y.(s-1))$, $\alpha.s = (V_{10}.(s+2),\ {}^{\wedge}T_9.s,\ {}^{\wedge}U_{24}.s,\ W.(s-2))$,

$\beta.s = ({}^{\wedge}\sigma.(s+3),\ {}^{\wedge}P_{40}.(s+5),\ R_{13}.s,\ Q_{10}.(s+3))$,

(s taken in Z_6 or Z_9).

A cyclable path in $\mathrm{RH}_{9,10}$ is given as follows:

($P_1.0$, $Q_1.0$, $S_3.6$, $T_{28}.1$, ${}^{\wedge}V_{13}3;.7$, ${}^{\wedge}T_9.5$, ${}^{\wedge}U_{24}.5$, $W.3$, $\alpha.4$, $\alpha.3$, $\alpha.8$, $\alpha.7$, $\alpha.6$, ${}^{\wedge}V_{10};9.7$, $\Sigma.0$, ${}^{\wedge}V_{14}.4$, ${}^{\wedge}V_{14}6;.7$, $\Sigma.1$, ${}^{\wedge}V_{14}.2$, $S_{16}.0$, $U_{59}.2$, $R_{13}.1$, $Q_{10}.4$, $\beta.2$, $\beta.3$, $\beta.7$, $\beta.5$, $\beta.0$, ${}^{\wedge}\sigma.4$, ${}^{\wedge}P_{40}26;.6$, ${}^{\wedge}R_5.8$, ${}^{\wedge}R_{18}.4$, ${}^{\wedge}T_{17}.2$, ${}^{\wedge}S_{27}.7$, ${}^{\wedge}V_9.3$, ${}^{\wedge}S_8.2$, ${}^{\wedge}U_{29}.2$, ${}^{\wedge}Q_6.7$, ${}^{\wedge}S_{12}.8$, ${}^{\wedge}S_{37}.1$, ${}^{\wedge}P_{41}.2$, ${}^{\wedge}T_3.0$, ${}^{\wedge}R_{11}.6$, ${}^{\wedge}S_{21}.4$, ${}^{\wedge}Q_{31}.2$, ${}^{\wedge}P_{32}.4$, ${}^{\wedge}U_{59}.1$, ${}^{\wedge}S_{12}.3$, ${}^{\wedge}S_{37}.5$, ${}^{\wedge}P_{41};26.6$).

3. A Hamilton cycle in $\mathrm{RG}_{7,9}$ and a simpler algorithm

If $j = i + 2$, then T_2 is free if and only if either $i \neq 2i'$, or $j \neq 2j'$, where $i', j' \in Z$. (Otherwise, for $i = 2i'$ and $j = 2j'$, there is a copy of $\mathrm{RG}_{i',j'}$ inside $\mathrm{RG}_{i,j}$ by means of the graph monomorphism $(F, f)\colon \mathrm{RG}_{i',j'} = (V'E') \to \mathrm{RG}_{i,j} = (V, E)$, given by $F(c) = (c, c)$ and accordingly for f).

For $RG_{7,9}$, 3-interspersion makes τ_0 independent from τ_3 and moc(16) = pmoc(16) = 4. This means that nonpalindromic vertices in $RJ_{7,9}$ appear in pairs $+A$, $-A$, such that $-A = \leftarrow(+A) = T_0(1, +A)$. For nonpalindromic vertices we always use a sign, while palindromic vertices do not have a sign (compare [3], Example $i = 7$).

As an example of a simpler algorithm due to Quintana than the one used to obtain the Hamilton cycle above we now construct a Hamilton cycle for the graph $RG_{7,9}$ by a slightly different procedure. We redefine $RH_{i,j} = RG/\tau_1, \tau_2$. Trying to find a cyclable path Ω starting off at $[11 \cdots 10 \cdots 0]$ and ending up at another convenient looped vertex in this redefined $RH_{i,j}$ amounts to be working leaving out rule (c) of back-writing for the induced adjacency. This will only affect the effort of lifting Ω to a Hamilton cycle in $RG_{i,j}$ as follows: Representing our graph as in the Example for $RG_{2,3}$ given above, the vertical links that completed a Hamilton cycle in $RG_{i,j}/\tau_2$ with the two corresponding zigzag liftings of Ω change their meaning. In fact $[11 \cdots 10 \cdots 0]$ was previously adjacent to $[1 \cdots 10 \cdots 00]$ by a vertical link and this was part of the desired Hamilton cycle. Now again we use the two zigzag paths but one of them is coordinate-rotated as follows: If $B = 1111111110000000$ in $V(RG_{7,9})$ assume that $[B]$ is in the top vertex part of $RH_{7,9}$ (following the pattern given for $RH_{2,3}$). Then the other end of the zigzag path $\Omega_{[B]}$ starting at $[B]$ is also in the top part. The other zigzag path $\Omega_{[A]}$ has its end vertices at the lower vertex part and one of them is $[A]$, where $A = 0000000001111111$. We lift $\Omega_{[B]}$ and $\Omega_{[A]}$ in all the possible ways to $RG_{7,9}$ but the additional edges lifting from vertical edges in $RG_{7,9}/\tau_2$ now are obtained differently. In fact B is adjacent not to A, the main representative of $[A]$, but to $T_2(7, A) = 1111111000000000$. Then the zigzag path Ω_B containing B remains as is and is joined through the edge $(B, T_2(7, A))$ to the zigzag path $T_2(7, \Omega_A)$, where Ω_A is the zigzag path containing A, to perform a spring path Ω_{AB}. We will show a cyclable path Ω in $RG_{7,9}$ containing the image of Ω_{AB}. The free end of $T_2(7, \Omega_A) \subseteq \Omega_{AB}$ in the lifting of Ω is adjacent to another representative of $[B]$. This allows the formation of a pattern having continuation by uniform coordinate rotation up to sixteen times. Finally the completion of the desired Hamilton cycle in $RG_{7,9}$ is obtained.

The graph $RJ_{7,9}/\tau_0$ determined by $r = 3$ and $\tau_3 : Z_4 \times RG_{7,9} \to RG_{7,9}$ has its set of vertices divisible in the following classes:

P One palindromic vertex P with oc = 1.

Q: Two nonpalindromic vertices Q with oc = 1.

R: Three palindromic vertices R with oc = 2.

S: Fourteen nonpalindromic vertices S with oc = 2.

T: Three nonpalindromic vertices with oc = 2 and $T_3(-1, T) = T_0(1, T)$.

U: Seven palindromic vertices U with oc = 4.

V: Seventy three nonpalindromic V with oc = 4.

W: Seven nonpalindromic W with oc = 4 and $T_3(-2, W) = T_0(1, W)$.

X: One nonpalindromic X with oc = 2 and $T_3(-1, X) = T_2(4, X)$.

All the representatives of V-vertices in $\mathrm{RH}_{7,9}/T_0$ form a path V in $\mathrm{RJ}_{7,9}$ with:

$\Phi(V) = [0101100111000111]$, $\pi(V) = \{(8, 9), (0, 2), (3, 4), (3, 10), (0, 10), (6, 8), (8, 14), (10, 11), (13, 15), (4, 8), (3, 8), (3, 13), (2, 7), (5, 15), (1, 3), (1, 14), (13, 15), (2, 7), (1, 11), (11, 15), (7, 14), (1, 13), (2, 13), (2, 3), (11, 15), (11, 13), (5, 6), (7, 8), (8, 13), (0, 5), (0, 2), (9, 14), (4, 11), (11, 13), (1, 13), (0, 12), (5, 7), (2, 6), (2, 10), (5, 10), (9, 14), (1, 2), (1, 6), (6, 14), (12, 14), (1, 9), (2, 3), (3, 14), (10, 11), (2, 4), (2, 3), (11, 13), (0, 1), (2, 10), (6, 11), (0, 15), (4, 5), (3, 8), (0, 14), (6, 11), (10, 11), (1, 11), (9, 11), (5, 12), (1, 6), (4, 10), (10, 15), (6, 10), (3, 13), (3, 11), (3, 8), (0, 13)\}$.

The path V is in fact a cycle. Moreover, V lifts to either cycles $\pm V.i$ in $\mathrm{RH}_{7,9}$, for $0 \leq i \leq 3$.

The following path Y, containing twenty three vertices, has its ends in class R, the first seven vertices after each end in class S and the other vertices in class W:

$\Phi(Y) = [1011111010010100]$, $\pi(Y) = \{(3, 11), (7, 15), (5, 13), (3, 11), (2, 8), (5, 13), (4, 6), (9, 10), (3, 15), (12, 14), (10, 0), (2, 8), (7, 11), (4, 6), (9, 14), (3, 11), (6, 8), (10, 15), (4, 12), (7, 8), (8, 13), (8, 0)\}$.

Class U forms its own path F:

$\Phi(U) = [1110101011100100]$, $\pi(U) = \{(1, 9), (3, 7), (2, 8), (12, 14), (4, 6), (11, 15)\}$.

The remaining vertices are going to be used to join liftings of the paths described above. They are:

$$Q_1 = [1111101001011000], \quad Q_2 = [1010111100001101],$$
$$T_1 = [1110110011100100], \quad T_2 = [1110110011001100],$$
$$T_3 = [1110110001101100], \quad P_1 = [1101010101010101],$$
$$X_1 = [1110100111001001], \quad R_3 = [1110101111000001].$$

The claimed cyclable path in $\mathrm{RH}_{7,9}$ is:

$(+V;65.0,\ U.2,\ (+V65;.0);31.0,\ +Y;20.1,\ R_3.1,\ +Y20;.1,\ (-Y;1.1)^{-1};1.1,\ (+V31;.0);18.0,\ +T_2.0,\ (+V;18.2)^{-1};8.2,\ (U.0)^{-1},\ (+V;65.2)^{-1},\ -V.2,\ (+V18;.2)^{-1},\ -T_3.0,\ +V18;.0,\ (-V18;.0)^{-1},\ +T_3.0,\ +V18;.3,\ (-V.3)^{-1},\ +V;65.3,\ U.1,\ (+V65;.3);31.3,\ +Y;20.0,\ R_3.0,\ (+Y20;.0);18.0,\ -T_1.0,\ (+V1;.0);15.0,\ P_1.0,\ (+V15;.0);7.0,\ -Q_1.0,\ -Q_2.0,\ (+V7;.0);1.0,\ X_1.0,\ +V1;.0,\ -X_1.0,\ ((-V;1.0)17;.0)^{-1},\ T_1.0,\ (-V;18.0)^{-1};1.0,\ (+V31;.3);18.3,\ -T_2.0,\ (-V;18.0)^{-1};49.0,\ +Q_2.0,\ -V24;.1,\ (+V65;.1)^{-1},\ (U.3)^{-1},\ (+V;65.1)^{-1},\ -V;24.1,\ +Q_1.0,\ (-V;24.0)^{-1})$.

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Note added in proof

We have learned recently that the Conjecture treated in this paper and attributed to Erdös appeared for the first time in written form in I. Hàvel, Semipaths in directed cubes, in Graphs and Other Combinatorial Topics, B.59, Teubner Texte zum Mathematik (Teubner, Leipzig) (3ième Symposium Tchecoslovaque de Th. graphes, Nov. 1982).

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A RESULT ON GENERALIZED LATIN RECTANGLES

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An alternative and simpler proof of the following result is given: Every $r \times s$ generalized partial latin rectangle Q on $A = \{1, 2, \ldots, k\}$ can be extended to an $n \times n$ generalized latin square on A if and only if $n \geq r + s - \min\{N(i) \mid i \in A\}$, where $N(i)$ denotes the number of times that the symbol i appears in Q.

1. Introduction

A *latin rectangle* of size $r \times s$ on a set of k symbols $1, 2, \ldots, k$ is an $r \times s$ matrix in which every entry is an integer between 1 and k and each integer occurs at most once in each row and each column. In [5], Ryser proved that every $r \times s$ latin rectangle Q can be extended to a $k \times k$ latin square on the same set of symbols $1, 2, \ldots, k$ if and only if each symbol occurs at least $r + s - k$ times in Q.

In this paper, we study a generalized version of latin rectangles which was studied by Andersen and Hilton [1–4]. In Theorem 2.3, we prove that every $m \times n$ generalized latin rectangle can be extended to form an $n \times n$ generalized latin square on the same set of symbols. We also prove that every $r \times s$ generalized partial latin rectangle Q on $A = \{1, 2, \ldots, k\}$ can be extended to an $n \times n$ generalized latin square on A if and only if $n \geq r + s - \min\{N(i) \mid i \in A\}$ where $N(i)$ denotes the number of times that symbol i appears in Q. Our proof is elementary and different from the one given by Hilton [4], who made use of de Werra's theorem on balanced edge-colouring of bipartite graph [6]. In the last section, we illustrate by an example the two different approaches.

2. Generalized latin rectangle

Definition 2.1. Let $L = (L_{ij})$ be an $m \times n$ $(m \leq n)$ matrix and $A = \{1, 2, \ldots, k\}$ satisfying the following:

(a) $L_{ij} \subseteq A$ for all $1 \leq i \leq m$, $1 \leq j \leq n$
(b) $\emptyset \neq \bigcup \{L_{ij} \mid j = 1, 2, \ldots, n\} \subseteq A$ for all $1 \leq i \leq m$
(c) $\emptyset \neq \bigcup \{L_{ij} \mid i = 1, 2, \ldots, m\} \subseteq A$ for all $1 \leq j \leq n$
(d) $L_{ip} \cap L_{iq} = \emptyset$ for all $p \neq q$, $1 \leq p$, $q \leq n$
(e) $L_{rj} \cap L_{sj} = \emptyset$ for all $r \neq s$, $1 \leq r$, $s \leq m$

Then $L=(L_{ij})$ is called a *generalized partial latin rectangle on* A. A generalized partial latin rectangle $L=(L_{ij})$ on A is called a *generalized latin rectangle on* A if

$$\text{for all } 1 \leq i \leq m, \qquad \bigcup \{L_{ij} \mid j=1,2,\ldots,n\} = A.$$

$L=(L_{ij})$ is called a *generalized latin square on* A if

(a) $L=(L_{ij})$ is a generalized latin rectangle on A

(b) $m=n$

(c) $\bigcup \{L_{ij} \mid i=1,2,\ldots,m\} = A$ for all $1 \leq j \leq n$

Definition 2.2. Let $L=(L_{ij})$ be an $m \times n$ $(m \leq n)$ generalized partial latin rectangle on A. An $r \times s$ $(r \geq m, s \geq n)$ generalized latin rectangle $L^*=(L^*_{ij})$ on B, $B \supseteq A$, is called an *extension* of L if

$$L^*_{ij} = L_{ij} \text{ for all } 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

If L has an extension L^*, then we say that L *can be extended* to L^*.

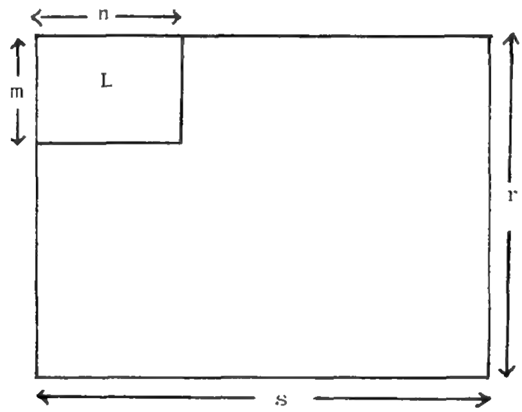


Diagram of L^*.

Theorem 2.3. *Every* $m \times n$ $(m \leq n)$ *generalized latin rectangle* $L=(L_{ij})$ *on* k *symbols* $1,2,\ldots,k$ *can be extended to form an* $n \times n$ *generalized latin square* $L^*=(L^*_{ij})$ *on the same set* $\{1,2,\ldots,k\}$.

Proof. Let $L=(L_{ij})$ be an $m \times n$ $(m \leq n)$ generalized latin rectangle on $\{1,2,\ldots,k\}$. Form a set of $m \times n$ (0, 1)-matrices $P_1, P_2, \ldots, P_k$ from $L=(L_{ij})$ as follows:

For all $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq t \leq k$

$$(i,j) \text{ entry of } P_t = \begin{cases} 1 & \text{if } t \in L_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

Since $L=(L_{ij})$ is a generalized latin rectangle, each symbol t $(t=1,2,\ldots,k)$ appears exactly once in each row and at most once in each column of L. It follows

that each P_t is an $m \times n$ (0, 1)-matrix with only one 1 in each row and at most one 1 in each column.

Augment each P_t into an $n \times n$ matrix P_t^* where (i, j) entry of

$$P_t^* = \begin{cases} (i, j) \text{ entry of } P_t & \text{for all } 1 \leq i \leq m,\ 1 \leq j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

It follows that each P_t^* has $(n-m)$ zero rows and $(n-m)$ zero columns. For each P_t^*, arbitrarily match each zero row to a zero column. Indicate the chosen matching by assigning value 1 to (i, j) entry of P_t^* if row i of P_t^* is matched to column j of P_t^*. Let $L^* = (L_{ij}^*)$ be an $n \times n$ matrix where $L_{ij}^* = \{t \mid (i, j) \text{ entry of } P_t^* \text{ is } 1,\ 1 \leq t \leq k\}$ for all $1 \leq i, j \leq n$. Then $L^* = (L_{ij}^*)$ is an $n \times n$ generalized latin square on $\{1, 2, \ldots, k\}$ since each P_t^* is an $n \times n$ (0, 1)-matrix with exactly one 1 in each row and each column. Furthermore, it is clear that (i, j) entry of L^* equals (i, j) entry of L for all $1 \leq i \leq m$, $1 \leq j \leq n$. Hence L^* is an extension of L. □

Corollary 2.4. *Every $m \times n$ $(m \leq n)$ latin rectangle $L = (L_{ij})$ on k symbols $1, 2, \ldots, k$ can be extended to form an $n \times n$ generalized latin square $L^* = (L_{ij}^*)$ on the same set $\{1, 2, \ldots, k\}$.*

Proof. Since every latin rectangle $L = (L_{ij})$ is a generalized latin rectangle with each L_{ij} $1 \leq i \leq m$, $1 \leq j \leq n$, being a singleton set, the result then follows from Theorem 2.3. □

Theorem 2.5. *Every $r \times s$ $(r \leq s)$ generalized partial latin rectangle $Q = (Q_{ij})$ on $A = \{1, 2, \ldots, k\}$ can be extended to an $n \times n$ generalized latin square $L^* = (L_{ij}^*)$ on A if and only if*

$$n \geq r + s - \min\{N(i) \mid i \in A\}$$

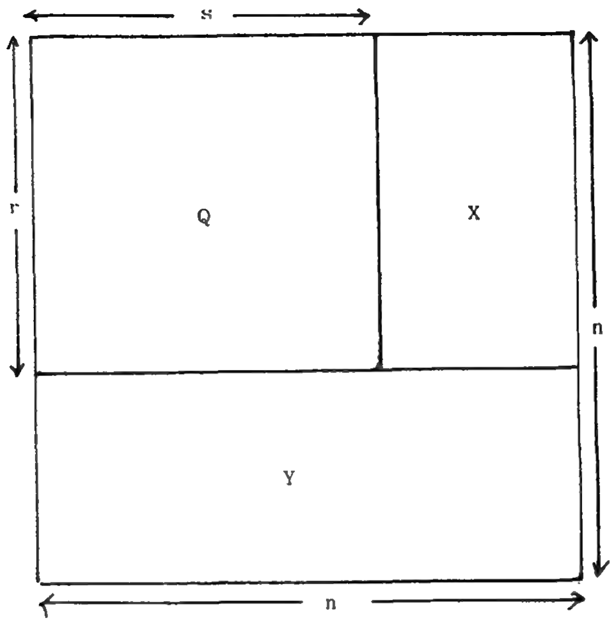
where $N(i)$ denotes the number of times that symbol i appears in Q.

Proof.

Necessity. Let $Q = (Q_{ij})$ be an $r \times s$ $(r \leq s)$ generalized partial latin rectangle on $A = \{1, 2, \ldots, k\}$. Suppose that $Q = (Q_{ij})$ can be extended to an $n \times n$ generalized latin square $L^* = (L_{ij}^*)$ on A, which is subdivided as shown on top of page 74.

It is required to prove that $n \geq r + s - \min\{N(i) \mid i \in A\}$. Suppose on the contrary that there exists $i \in A$ such that $n < r + s - N(i)$. This implies that $r - N(i) > n - s$.

Since $r - N(i)$ is the number of rows that symbol i does not appear in Q and $L = Q + X$ is a generalized latin rectangle, X must have at least $r - N(i)$ columns. But the number of columns in X is $n - s$ and $n - s < r - N(i)$, a contradiction. Hence $n \geq r + s - \min\{N(i) \mid A\}$.



Sufficiency. Let $Q = (Q_{ij})$ be an $r \times s$ $(r \leqslant s)$ generalized partial latin rectangle on A. Let $n_1 = r + s - \min\{N(i) \mid i \in A\}$ and $n \geqslant n_1$.

Case 1: $n = n_1$

Let $M_i = A - \bigcup \{Q_{ij} \mid j = 1, 2, \ldots, s\}$ for all $1 \leqslant i \leqslant r$. Let $L = (L_{ij})$ be an $r \times n$ matrix defined as follows:

(i) For all $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, $L_{ij} = Q_{ij}$

(ii) $L_{1(s+1)} = M_1$, $L_{1q} = \emptyset$ for all $s + 2 \leqslant q \leqslant n$

(iii) For all $2 \leqslant i \leqslant r$, $s + 1 \leqslant j \leqslant n$, $t \in L_{ij} \Leftrightarrow t \in M_i$ and j is the least integer in $\{s + 1, \ldots, n\}$ such that

$$t \notin \bigcup \{L_{pj} \mid p = 1, 2, \ldots, i - 1\}.$$

Then $L = (L_{ij})$ is an $r \times n$ generalized latin rectangle on A. By Theorem 2.3, L can be extended to form an $n \times n$ generalized latin square $L^* = (L^*_{ij})$ on A.

Case 2: $n > n_1$

By Case 1, we can extend Q to an $n_1 \times n_1$ generalized latin square L^* on A. Let B be an $(n - n_1) \times (n - n_1)$ generalized latin square with each diagonal entry equal to A and empty elsewhere.

Then

$$\left[\begin{array}{c|c} L^* & 0 \\ \hline 0 & B \end{array}\right]$$

is the required $n \times n$ generalized latin square on A. $\square$

Theorem 2.5 can be reformulated as follows:

Theorem 2.6 (Hilton [4], Theorem 2). *Every $r \times s$ ($r \leqslant s$) generalized partial latin rectangle $Q = (Q_{ij})$ on $A = \{1, 2, \ldots, k\}$ can be extended to an $n \times n$ generalized latin square $L^* = (L^*_{ij})$ on A if and only if*

$$N(i) \geqslant r + s - n \qquad \textit{for all } i \in A$$

where $N(i)$ denotes the number of times that symbol i appears in Q.

3. Example

Consider the 2×3 generalized partial latin rectangle $Q = (Q_{ij})$ on $A = \{1, 2, 3, 4, 5\}$ as shown below:

1	2	3 4
3	1 4	

We shall illustrate how $Q = (Q_{ij})$ can be extended to a 5×5 generalized latin square $L^* = (L^*_{ij})$ on A by two different approaches, namely

(a) the approach used in the proof of Theorem 2.5

(b) the approach used by Hilton [4] based on de Werra's theorem [6].

Approach (a). In what follows, we shall use the same notations as in the proof of

Theorem 2.5.

$$M_1 = \{5\}$$
$$M_2 = \{2, 5\}$$

$L = (L_{ij})$ is a 2×5 matrix with

$$L_{ij} = Q_{ij} \quad \text{for all } 1 \leq i \leq 2, \quad 1 \leq j \leq 3$$
$$L_{14} = \{5\}$$
$$L_{24} = \{2\}$$

and

$$L_{25} = \{5\}$$

1	2	3 4	5	
3	1 4		2	5

Diagram of L

Hence, L is a 2×5 generalized latin rectangle on A. We then form a set of 5×5 (0, 1)-matrices $P_1^*, \ldots, P_5^*$ as obtained in the proof of Theorem 2.3.

$$\underset{P_1^*}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}} \underset{P_2^*}{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}} \underset{P_3^*}{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\underset{P_4^*}{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}} \underset{P_5^*}{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}}$$

Then

1	2	3 4	5	
3	1 4		2	5
2 4 5	3	1		
	5	2	1 3 4	
		5		1 2 3 4

is a 5×5 generalized latin square on A.

Approach (b). In the proof of Theorem 2.6, Hilton [4] made use of the following theorem:

Theorem (de Werra) [6]. *For each $k \geq 1$, any finite bipartite graph has a balanced edge-coloring with k colors.*

Step 1: *Extend Q to a 2×5 generalized latin rectangle L on A.*

Form a bipartite graph $G(X, Y)$ as follows:

The vertices in X correspond to the rows of Q, say $X = \{R_1, R_2\}$. The vertices in Y correspond to the symbols in A, say $Y = \{1, 2, 3, 4, 5\}$. R_i $(1 \leq i \leq 2)$ is joined to y $(1 \leq y \leq 5)$ in Y if and only if y does not occur in row i of Q.
Since the degree of each vertex in Y is less than or equal to two, by de Werra's theorem, G has an equitable edge-coloring with two colors, say color 4 and color 5 (for simplicity we use colors 4 and 5 to indicate the new columns 4 and 5). Place symbol y into entry (i, j) of L if and only if there is an edge (R_i, y) with color j.

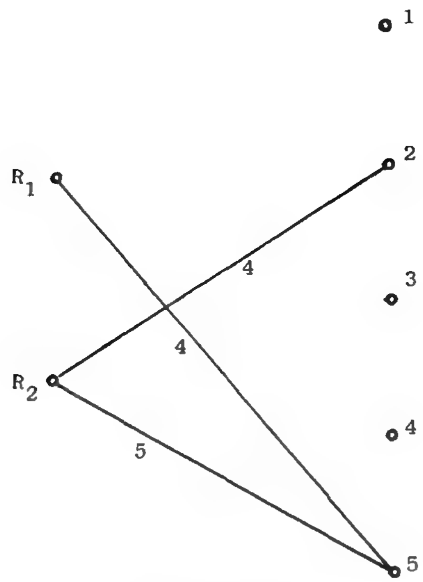


Diagram of $G(X, Y)$

We thus obtain the following generalized latin rectangle L:

1	2	3 4	5	
3	1 4		2	5

Diagram of L

Step 2: *Extend L to a* 5×5 *generalized latin square* L^* *on* A.

Form a bipartite graph $G^*(X^*, Y^*)$ where vertices of X^* correspond to columns of L and vertices of Y^* correspond to symbols in A, say $X^* = \{c_1, c_2, c_3, c_4, c_5\}$ and $Y^* = \{1, 2, 3, 4, 5\}$. $c_i \in X^*$ is joined to $y \in Y^*$ if and only if y does not occur in column i of L.

As in Step 1, we can give G^* an equitable edge-coloring with 3 colors, namely colors 3, 4 and 5.

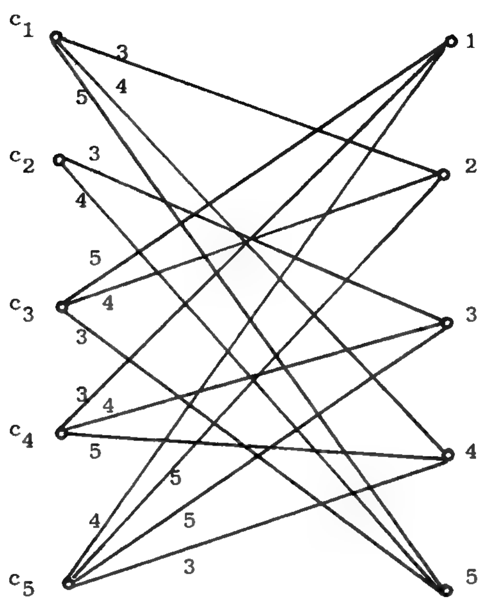


Diagram of $G^*(X^*, Y^*)$

With this coloring, we obtain L^* as follows:

1	2	3 4	5	
3	1 4		2	5
2	3	5	1	4
4	5	2	3	1
5		1	4	3 2

Diagram of L^*

Remark. As illustrated from the above example, it is clear that our approach in extending a generalized partial latin rectangle to a generalized latin square is much simpler than the one given by Hilton [4], especially when the corresponding bipartite graph has large maximum degree.

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North-Holland

PROBLEMS AND RESULTS IN COMBINATORIAL ANALYSIS AND GRAPH THEORY

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I wrote many papers with this and similar titles. In my lecture I stated several of my old solved and unsolved problems some of which have already been published elsewhere. To avoid overlap as much as possible, I state here only relatively new problems.

First I state two recent problems of Nesetril and myself.

1. Let G be a graph each vertex of which has degree not exceeding n. It is true that if our G has more than $5n^2/4$ edges then G contains two strongly independent edges? (i.e. two edges, which are vertex disjoint and for which the subgraph of G induced by the vertices of these two edges contains only these two edges).

Very recently Fan Chung and Trotter and independently and simultaneously Gyárfás and Tuza proved this conjecture. The proofs are quite complicated. It is easy to see that the result is best possible.

We also formulated the following much more difficult and interesting Vizing type conjecture: Let G be a graph each vertex of which has degree not exceeding n. Is it then true that G is the union of at most $5n^2/4$ sets of strongly independent edges? If this conjecture fails one can try to determine the smallest $f(n)$ so that every G each vertex of which has degree $\leqslant n$ is the union of $f(n)$ sets of strongly independent edges. $f(n) < 2n^2$ is easy.

One could perhaps try to determine the smallest integer $h_r(n)$ so that every G of $h_r(n)$ edges each vertex of which has degree $\leqslant n$ contains two edges so that the shortest path joining these edges has length $\geqslant r$. The order of magnitude of $h_r(n)$ is easily seen to be n^{r+1} but the exact value of $h_r(n)$ is unknown. This problem seems to be interesting only if there is a nice expression for $h_r(n)$.

Our second problem states as follows: Let $G(n)$ be a graph of n vertices $x_1, \ldots, x_n$. A subset $x_{i_1}, \ldots, x_{i_r}$ is said to be a minimal cut if the omission of these vertices disconnects $G(n)$, but no subset $x_{i_1}, \ldots, x_{i_r}$ disconnects $G(n)$. Denote by $c(n)$ the maximal number of minimal cuts a $G(n)$ can have. Seymour observed $c(3m+2) \geqslant 3^m$. To see this let $G(3m+2)$ have the vertices x, y and there be m independent paths of length 4 joining x and y. Perhaps $c(3m+2) = 3^m$, we could not even prove that $c(n)^{1/n} \to \alpha < 2$.

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2. Gallai conjectured more than a year ago that if $G(n)$ is a graph which contains no wheel (i.e. a cycle and a vertex joined to all the vertices of our cycle) then $G(n)$ contains at most $\frac{1}{8}n^2$ triangles. It is easy to see that this conjecture if true is best possible. Let $|A| = \frac{1}{2}n$, $|B| = [\frac{1}{2}(n+1)]$, join every vertex of A to every vertex of B and add in B a matching. It is well known and not hard to see that every graph of $\frac{1}{4}n^2 + \frac{1}{4}n + 1$ edges contains a wheel (i.e. our graph is the largest graph which has no wheel). Unfortunately this does not seem to help with Gallai's conjecture.

3. Recently Gallai and I posed the following problem: Denote by $h(n)$ the smallest integer so that every $G(n)$ has a set of $\leq h(n)$ vertices $x_1, \ldots, x_t$, for which every clique of $G(n)$ contains at least one of these x_i's. It is easy to see that $h(n) \leq n - \sqrt{n}$. We conjecture that $h(n)$ equals to the smallest integer for which every graph of n vertices which has no triangles has a set of at least $n - h(n)$ independent vertices. To convince the reader that our conjecture is not unreasonable consider the set of all graphs on n vertices which have no triangle. Such a graph must have at least $n - h(n)$ independent vertices and there is such a graph $G_1(n)$ which has no triangle and which has exactly $n - h(n)$ independent vertices. Thus to represent all cliques (i.e. in this case all edges) of our graph we need $h(n)$ vertices (namely the complement of our largest independent set). Thus it was perhaps not unreasonble to assume that $h(n)$ vertices will always suffice to represent all cliques of our graph. We made no progress with this conjecture which is perhaps completely wrongheaded. We could not make any progress even if we assumed that our $G(n)$ has no $K(4)$. In this case we only would have to represent all $K(3)$'s of $G(n)$ and all the edges not contained in a $K(3)$.

Gallai further conjectured that if $G(n)$ is a chordal graph (i.e. all cycles C_n, $n > 3$ have a diagonal) then all cliques can be represented by $[\frac{1}{2}n]$ vertices. This conjecture was indeed proved by Aigner, Andreae and Tuza.

4. The problem of Gallai and myself naturally leads to the problem of Ramsey numbers. Many papers on these questions were published and to avoid repetition I state here only a few of them and will give an admittedly incomplete list of references.

Denote by $r(u, v)$ the smallest integer so that every graph on $r(u, v)$ vertices either contains a complete graph of u vertices or an independent set of v vertices. It is more usual to use the following (equivalent) definition: $r(u, v)$ is the smallest integer so that if we color the edges of $K(r(u, v))$ (i.e. the complete graph of $r(u, v)$ vertices) by two colors I and II then there is either a $K(u)$ all whose edges have color I or a $K(v)$ all whose edges have color II. $r(n, n)$ is the diagonal Ramsey number, it is the smallest integer for which the every complete graph of $r(n, n)$ vertices whose edges are coloured by two colors always contains a monochromatic $K(n)$. $r(3, 3) = 6$, $r(4, 4) = 18$ (this is an old result of Greenwood

and Gleason) and $r(5, 5)$ is unknown. The best current bounds are

$$c_1 n 2^{n/2} < r(n, n) < \binom{2n}{n} \Big/ (\log n)^{\varepsilon}. \tag{1}$$

I proved the lower bound in (1) by probabilistic methods. The value of the constant was improved by Joel Spencer. The upper bound in (1) was recently obtained by Rödl and is not yet published. I offer 100 dollars for a proof that

$$\lim_{n\to\infty} r(n, n)^{1/n} = c \tag{2}$$

exists and I offer 10 000 dollars for a disproof. I am of course sure that (2) holds. I offer 250 dollars for the determination of c. $\sqrt{2} \leq c \leq 4$ follows from (1), perhaps $c = 2$? Let us now give a very short discussion of the non-diagonal Ramsey numbers. We have

$$\frac{c_1 n^2}{(\log n)^2} < r(3, n) < \frac{c_2 n^2}{\log n}. \tag{3}$$

The lower bound in (3) is due to me, the upper bound is due to Ajtai, Komlós and Szemerédi, who improved by a factor $\log \log n$ the previous result of Graver and Yackel. It is perhaps not hopeless to try to get an asymptotic formula for $r(3, n)$.

It would be reasonable to guess that for every fixed k and $\varepsilon > 0$ if $n > n_0(\varepsilon, k)$

$$r(k, n) > n^{k-1-\varepsilon} \tag{4}$$

but the proof of (4) presented so far unsurmountable difficulties, even for $k = 4$. At first I thought that the difficulties are only technical and the probability method will give (4), but perhaps I was too optimistic.

The best constructive lower bound for $r(n, n)$ is due to Peter Frankl, who proved

$$r(n, n) > \exp\left(\frac{c(\log n)^2}{\log \log n}\right).$$

I offer 100 dollars for a constructive proof of $r(n, n) > (1 + c)^n$. I am afraid that there are easier methods of earning 100 dollars.

Several of us tried to prove simple inequalities between Ramsey numbers. We all failed so far. The main difficulty is perhaps the lack of constructive methods. Here is a sample which shows our ignorance:

Is it true that

$$r(n + 1, n) - r(n, n) > cn^2. \tag{5}$$

'Clearly' (?).

$$\lim r(n + 1, n)/r(n, n) = C^{\frac{1}{2}} \quad \text{where} \quad r(n, n)^{1/n} \to C. \tag{6}$$

(6) seems quite hopeless at present. V.T. Sós and I failed to prove

$$\frac{r(3, n+1) - r(3, n)}{n} \to 0 \quad \text{and} \quad r(3, n+1) - r(3, n) \to \infty. \tag{7}$$

The second inequality in (7) should be perhaps easier than the first. Simonovits and I tried unsuccessfully to prove that for every $k \geqslant 4$

$$\lim_{n\to\infty} r(k+1, n)/r(k, n) = \infty \tag{8}$$

(4) is easy for $k = 3$.

I just mention one of two problems on generalized Ramsey numbers. $r(n; \mathscr{G})$ is the smallest integer t, so that if we color the edges of $K(t)$ by two colors then either color I contains a $K(n)$ or color II contains G. It is particularly frustrating that (C_4 is a cycle of length four)

$$\lim_{n\to\infty} r(n, K(3))/r(n, C_4) = \infty \tag{9}$$

has not yet been proved. (9) is an old conjecture of mine. I in fact conjectured that the following much sharper (and much more doubtful) result holds:

$$r(n; C_4) < n^{2-\varepsilon} \text{ for some } \varepsilon > 0 \quad \text{and} \quad n > n_0(\varepsilon).$$

Szemerédi proved (unpublished)

$$r(n; C_4) < cn^2/(\log n)^2. \tag{10}$$

(10) in view of (3) 'nearly' proves (9). To end this chapter I state an old and nearly forgotten conjecture of Bondy and myself: Let n be odd. Color the edge of a $K(4n-3)$ by three colors. Then there always is a monochromatic C_n. The analogous conjecture for two colors was proved by V. Rosta and Faudree and Schelp.

Several excellent survey papers on Ramsey numbers were written by Burr and Rosta, see also a forthcoming book on this subject by Burr, Faudree and Schelp. Faudree, Rousseau, Schelp and I have many papers on this subject.

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5. During my last visit to Memphis State University Faudreee, Rousseau, Schelp and I came across the following nice problem which to our surprise is

perhaps difficult. (The problem came up in connection of our work on generalised Ramsey theory, but as such is quite independent of it.) Is it true that there is an absolute constant $c>0$ so that every $\mathcal{G}(n;2n-1)$ has a subgraph $G(m)$, $m<n(1-c)$ so that every vertex of $G(m)$ has degree ≥ 3? Faudree could prove this with $m \leq n - c\sqrt{n}$ instead of $n(1-c)$. C_{n-1} and an nth vertex joined to every vertex of our C_{n-1} shows that $2n-1$ cannot be replaced by $2n-2$.

Pósa and I proved that every $\mathcal{G}(n;n+k)$ contains a cycle of size not exceeding $c_1 n \log k/k$ and apart from the value of c_1 this is best possible. It might be of some interest to try to obtain the exact size $n \cdot g(k)$ of this cycle for small values of k, for example $g(1)=\frac{2}{3}$, $g(5)=\frac{1}{3}$. I believe we determined $g(k)$ for $k \leq 5$, for larger values of k the exact determination of $g(k)$ gets more and more laborious and tricky. A cycle can of course be considered as a subgraph of degree 2, but perhaps our old result with Pósa throws no light on our conjecture, since it is not difficult to prove that for every c_1 there is a c_2 so that there is a $\mathcal{G}(n;c_1 n)$ for which every subgraph $\mathcal{G}(m)$ each vertex of which has degree ≥ 3 has more than $c_2 n$ vertices. The existence of a such a graph follows easily by the probability method, but a direct construction will perhaps also be easy. We have not determined the exact dependence of c_2 on c_1.

Reference

[1] P. Erdős and L. Pósa, On the maximal number of disjoint circuits of a graph, Publ. Math. Debrecen 9 (1962) 3–12.

6. Several years ago Sauer and I asked the following question: Let $f_k(n)$ be the smallest integer for which every $G(n;f_k(n))$ contains a regular subgraph of degree k. Trivially $f_2(n)=n$, but we could not get any non-trivial results for $f_k(n)$ for $k=3$. In particular we could not prove $f_3(n)/n \to \infty$ and $f_3(n)<n^{1+\epsilon}$. A few months ago Pyber proved

$$f_k(n) < c_2 k^2 n \log n \tag{1}$$

Pyber, Rödl and Szemerédi proved

$$cn \log\log n < f_3(n). \tag{1'}$$

Their proof of both the upper and lower bound of (1) is ingenious. It would be nice to improve (1) further and get an asymptotic formula for $f_3(n)$ and generally, $f_k(n)$.

Szemerédi once asked: Denote by $F_k(n)$ the smallest integer so that every $\mathcal{G}(n;F_k(n))$ contains an induced subgraph of degree k. How large is $F_k(n)$? Again it is trivial that $F_2(n)=n$. I observed that $F_3(n)<cn^{\frac{5}{3}}$ since it is easy to see that every $\mathcal{G}(n;cn^{\frac{5}{3}})$ contains either a $K(4)$ or an induced $K(3, 3)$. It would be nice to improve this if possible.

Reference

[1] L. Pyber, Regular subgraphs of dense graphs, Combinatorica 5 (1985) 347–349.

7. Now I discuss some problems on extremal graph theory. Let G be any graph, denote by $T(n; G)$ the Turán number of G, i.e. the smallest integer so that every $\mathcal{G}(n; T(n; G))$ contains G as a subgraph. Many papers on this subject have been published recently (some by myself). Bollobás published an excellent book on this subject and Simonovits an excellent survey paper; thus to avoid repetitions I will try to mention as much as possible only new problems or questions which have been neglected.

In a paper Simonovits and I investigated the following question: Let $\mathcal{G}(n; T(n; G) + \ell)$ be a graph. How many copies of G must our graph contain? For large ℓ we get very satisfactory results but we had little success for small values of ℓ. In particular for which graphs G it is true that every $\mathcal{G}(n; T(n; \mathcal{G}))$ contains two (or more generally) many copies of G. If $\mathcal{G}$ is a triangle then Rademacher, almost immediately after he heard of the result of Turán $(T(n; K(3)) = [\frac{1}{4}n^2] + 1)$, proved that every $G(n; [\frac{1}{4}n^2] + 1)$ contains at least $[\frac{1}{2}n]$ triangles and this result is best possible. This result was extended to $G(n; [\frac{1}{4}n^2] + \ell)$ first for small values of g by me and later to a much larger range by Bollobás, Lovász and Simonovits. On the other hand Simonovits and I conjectured that $G(n; T(n; C_4))$ contains $c_1 n^{\frac{1}{2}}$ C_4's. If true this result is best possible, but we could not even prove that it contains 2 C_4's. Can one characterise those graphs G for which every $G_1(n; T(n; G))$ must necessarily contain at least two subgraphs isomorphic to G?

Here I would like to insert one problem on hypergraphs which perhaps will lead to interesting problems. I only state the simplest case. A classical problem of Turán states: Let $T^{(3)}(n; K^{(3)}(4))$ be the smallest integer so that every triple system on n elements and $T^{(3)}(n; K^{(3)}(4))$ triples contains a $K^{(3)}(4)$, i.e. a set of 4 elements all whose triples are in our system. The determination of $T^{(3)}(n; K^{(3)}(4))$ seems to be very difficult. Is it true that such a triple system must contain at least two (and perhaps in fact cn) $K^{(3)}(4)$'s. Observe that it is easy to see that every $G(n; [\frac{1}{4}n^2] + 1)$ contains an edge e and $c_1 n$ other vertices $X_1, \ldots, X_t$ so that all these vertices form a triangle with e in our $\mathcal{G}(n; [\frac{1}{4}n^2] + 1)$. Bollobás and I conjectured and Edwards proved that $c_1 = \frac{1}{6}$ is best possible. Is it true that every 3-uniform hypergraph (or triple system) $G^{(3)}(n; T^{(3)}(n; K^{(3)}(4))$ contains an edge e and $c_1 n$ vertices $x_1, \ldots, x_t$ so that e and x_i are a $K^{(3)}(4)$ in our hypergraph? This problem is of course open even for $t = 2$. It follows easily from our results with Simonovits that a $G^{(3)}(n; (1 + \varepsilon)T(n; K^{(3)}(4)))$ contains such a system.

References

[1] B. Bollobás, Relations between sets of complete subgraphs, Fifth British Combinatorial Conference, Utilitas Mathematicae (Winnipeg, 1976) 161–170.

[2] P. Erdős and M. Simonovits, Supersaturated graphs and hypergraphs, Combinatorica 3 (1983) 181–192.
[3] L. Lovász and M. Simonovits, On the number of complete subgraphs of a graph, Fifth British Combinatorial Conference, Utilitas Mathematicae (Winnipeg, 1976) 431–441.

8. Now I return to ordinary graphs, i.e. $(r=2)$. The results of Stone, Simonovits and myself show that the most interesting open problems are if G is bipartite. I first state some of our favourite conjectures with Simonovits.

Is it true that for every bipartite G there is a rational $\alpha_1 = \alpha(G_1) \leqslant \alpha < 2$ for which

$$\lim_{n\to\infty} T(n; G)/n^{\alpha} = c(G), \quad 0 < c(G) < \infty \tag{1}$$

exists? Is it true that for every rational α, $1 \leqslant \alpha \leqslant 2$ there is a bipartite G for which the limit (1) exists? At present these conjectures which are about 20 years old are beyond our reach, and in fact we have no real evidence for their truth. We have no idea of the possible value of $c(G)$, it would perhaps be reasonable to assume that $c(G)$ is algebraic.

Let G be bipartite. Denote by $K(G) = r$ the largest integer for which G has an induced subgraph G' each vertex of which has degree $\geqslant r$. We conjectured that then

$$2 - \frac{1}{r-1} < \alpha(G) \leqslant 2 - \frac{1}{r}. \tag{2}$$

Thus in particular if $r = 2$, i.e. if G has no induced subgraph each vertex of which has degree >2, then

$$T(n; G) < cn^{\frac{3}{2}}. \tag{3}$$

On the other hand if G has an induced subgraph each vertex of which has degree $\geqslant 3$ then our conjecture would imply $\alpha(G) > \frac{3}{2}$. There is some (admittedly inadequate) evidence for our conjectures. Let G be the graph defined by the edges of the three-dimensional cube. We proved

$$T(n; G) < cn^{\frac{8}{5}} \tag{4}$$

and we believe $\alpha(\mathcal{G}) = \frac{8}{5}$, but unfortunately we could not even prove $\alpha(\mathcal{G}) > \frac{3}{2}$. In fact even $T(n; G)/n^{\frac{3}{2}} \to \infty$ is open. We proved that $T(n; G - e) < cn^{\frac{3}{2}}$. Further we have

$$T(n; K(r, r) - e) < cn^{2-1/(r-1)}.$$

A nice test case of our conjecture is the following: Let G_t be a graph of $1 + t + \binom{t}{2}$ vertices and $t + t(t-1)$ edges defined as follows:

The vertices of G_t are x; $y_1, y_2, \ldots, y_t$, $z_1, \ldots, z_{\binom{t}{2}}$. x is joined to all the y's and each z is joined to two of the y's so that every pair (y_i, y_j) is joined to exactly one z. Our conjecture (3) would imply

$$T(n; G_t) < c_t n^{\frac{3}{2}}. \tag{5}$$

For $t=3$ (5) is easy and we have no proof for $t \geqslant 4$. Omit X from G_t then we obtain G_t'. G_3' is C_6 and we know $\alpha(G_3') \approx \frac{4}{3}$. Faudree and Simonovits proved $\alpha(\mathcal{G}_4') < \frac{3}{2}$. Perhaps for every t $\alpha(\mathcal{G}_t') < \frac{3}{2}$. Unfortunately their ingenious proof only works for $t=4$.

Let now G_i, $1 \leqslant i \leqslant r$ be a family of graphs. Define $T(n; \mathcal{G}_1, \ldots, \mathcal{G}_r)$ as the smallest integer for which every $\mathcal{G}(n; T_n(\mathcal{G}_1, \ldots, \mathcal{G}_r))$ contains one of the G_i''s as a subgraph. Simonovits and I asked if there is a system $G_1, \ldots, G_r$ for which

$$\lim_{n\to\infty} T(n; G_1, \ldots, G_r) / \min_{i=1,2,\ldots,r} T_n(\mathcal{G}_i) = 0? \tag{6}$$

Perhaps our conjecture (1) can be generalised and there is an $\alpha(\mathcal{G}_1, \ldots, \mathcal{G}_r)$ for which

$$\lim_{n\to\infty} T(n; \mathcal{G}_1, \ldots, \mathcal{G}_r)/n^{\alpha(\mathcal{G}_1,\ldots,\mathcal{G}_r)} = c, \quad 0 < c < \infty \tag{7}$$

Faudree and Simonovits believe that

$$\alpha(C_4, \mathcal{G}_4') < \min(\alpha(C_4), \alpha(\mathcal{G}_4')) \tag{8}$$

and they hope that their method will give (8).

To end this long chapter I make a few remarks on some questions which certainly have not been investigated carefully. Let ℓ_k be the largest integer for which there is a $\mathcal{G}(k; \ell_k)$ satisfying

$$\lim_{n=} T(n; \mathcal{G}(k; \ell_k)/n^{\frac{3}{2}} < \infty. \tag{9}$$

$K(2; k-2)$ shows that $\ell_k \geqslant 2k-4$ and our conjecture (3) easily gives $\ell_k = 2k-4$. Perhaps this can be proved without (3) but by the probability method I could only prove that $T(n; G(k; 2k-3)) > cn^{\frac{3}{2}}$. Perhaps the following problem is more interesting: What is the largest integer t_k for which there is a $G(k; t_k)$ satisfying

$$T(n; \mathcal{G}(k; t_k)) = o(n^{\frac{3}{2}})? \tag{10}$$

If (10) holds then $\mathcal{G}(k; t)$ can of course not contain a C_4. It would be interesting to determine other forbidden bipartite graphs whose presence prevents (10) from holding. I have no good guess about the size of t_k, perhaps

$$2k - c_1 k^{\frac{1}{2}} < t_k < 2k - c_2 k^{\frac{1}{2}}. \tag{11}$$

References

[1] B. Bollobás, Extremal graph theory, London Math. Soc. Monographs N.11 (Academic Press, 1978).

[2] R.T. Faudree and M. Simonovits: On a class of degenerate extremal problems, Combinatorica 3 (1) (1983) 83–93.

[3] M. Simonovits, Extremal graph problems, degenerate extremal problems, and supersaturated

graphs, Progress in graph theory, Silver Jubilee Conference Combinatorics (Univ. of Waterloo Acad. Press 1984) 919–937.

[4] M. Simonovits, Estremal graph theory, Selected topics in graph theory II L.W. Beineke and R.J. Wilson, Eds (Academic Press, London, New York) 419–437.

9. Now I give a very short discussion of extremal problems on hypergraphs. To make the paper short I state only a few new problems. A long paper of mine is in the press on similarities and differences of extremal problems between graphs and hypergraphs and Frankl and Füredi have a long forthcoming paper in the J. Combin. Theory (A) on this subject.

Let $G_1^{(3)}$ be two triangles with a common edge and $G_2^{(3)}$ be the following $G^{(3)}(6;3)$ having the vertices x_1, x_2, x_3, x_4, x_5, x_6 and the edges $\{x_1, x_2, x_3\}\{x_3, x_4, x_5\}\{x_2, x_4, x_6\}$. An old problem of W. Brown, V.T. Sós and myself asked for the determination or estimation of $T^{(3)}(n; G_1^{(3)}, G_2^{(3)})$. Ruzsa and Szemerédi proved that

$$T^{(3)}(n; G_1^{(3)}, G_2^{(3)})/n^2 \to 0 \tag{1}$$

but for every $\varepsilon > 0$ $T^{(3)}(n; G_1^{(3)}, G_2^{(3)})/n^{2-\varepsilon} \to \infty$. In fact they prove a sharper result. An asymptotic formula for $T^{(3)}(n; G_1^{(3)}, G_2^{(3)})$ seems hopeless at present. (1) is certainly a new phenomenon. I then asked if there exists for some r a $G^{(r)}$ so that there is an α for which

$$T(n; G^{(r)})/n^{\alpha} \to 0$$

but for every $\varepsilon > 0$ $T(n; G^{(r)})/n^{\alpha-\varepsilon} \to \infty$. Frankl and Füredi found such a $G^{(r)}$ for $r = 5$, $\alpha = 4$. It is not yet known if for $r < 5$ such a hypergraph exists.

Frankl gave a talk on hypergraphs at our meeting in Hakone. After his lecture I asked: Is it true that every $G^{(3)}(n; 1 + \binom{n+1}{2})$ contains our $G_2^{(3)}$ and that the only $G^{(3)}(n; \binom{n-1}{2})$ which does not contain our $G_2^{(3)}$ consists of the $\binom{n-1}{2}$ triples which have a common vertex? Frankl informed me that this has already been proved by him and Füredi but that the proof is not quite simple. I at first though that if every vertex is contained in only $o(n^2)$ triples then every such $G^{(3)}(n; \varepsilon n^2)$ will contain our $G_2^{(3)}$. This was easily disproved by Frankl but perhaps such a $G^3(n)$ can only have $\frac{1}{2}n^2(1 + o(1))$ edges. I then asked: Is it true that if every pair of vertices (x, y) is contained in only $< Cn^{\frac{1}{2}}$ triples then every $G^{(3)}(n; \varepsilon n^2)$ must contain our $G_2^{(3)}$? During our excursion Füredi found the following nice counterexample:

Let $n = p^2 + p + 1$. We will have $2n$ elements A and L where A corresponds to the points and L to the lines of a finite geometry of n points. We divide A into two disjoint sets $B \cup C$, both having $(\frac{1}{2} + o(1))p^2$ elements and both meet every line of our finite geometry in $(\frac{1}{2} + o(1))\frac{1}{2}n^{\frac{1}{2}}$ points, Füredi's system now consists of the $(1 + o(1))\frac{1}{4}n^2$ triples (x, y, ℓ) where $x \in B$, $y \in C$ and $\ell \in L$ where x and y are on the line ℓ.

Whereupon I modified my conjecture:

Assume that every (x, y) is contained in only $o(n^{\frac{1}{2}})$ triples of our system. Then if such a triple system has cn^2 edges must it contain a $G_2^{(3)}$? It seems to speak against this conjecture that it was born as a response to several counterexamples. To end this section I would like to state an old problem of mine which seems very difficult: Is it true that for every k and $\varepsilon > 0$ there is an n_0 so that for every $n > n_0$ every $G^{(3)}(n; \varepsilon n^2)$ contains either a $G_2^{(3)}$ or a $G^{(3)}(k; k+3)$? For $k = 3$ this was our problem with Brown and V.T. Sós which was settled by Ruzsa and Szemerédi but for $k > 3$ very serious new difficulties appear and the problem is still very much open.

By the way Frankl proved a result on hypergraphs which is related to conjecture (1) of the previous chapter.

Reference

[1] P. Frankl, All rationals occur as exponents, J. Combin. Theory (A) 42 (1986) 200–206.

10. To end the paper I state a few miscellaneous problems. First of all here is a very nice problem of Tuza. Let $\mathcal{G}$ be a graph and k the largest integer for which G has k edge disjoint triangles. Is it then true that G can be made triangle free by the omission of at most $2k$ edges? $K(4)$ and $K(5)$ shows that if true the result is best possible. If true many generalisations and extensions will be possible.

Rothschild and I posed a few years ago the following problem: Assume that every edge of a $\mathcal{G}(n; cn^2)$ is contained in a triangle. Denote by $h(n; c)$ the largest integer so that every such graph has an edge which is contained in at least $h(n; c)$ triangles. The determination or good estimation of $h(n; c)$ does not seem to be quite easy. Szemerédi observed that his Regularity Lemma easily gives that for every $c > 0$

$$\lim_{n\to\infty} h(n; c) = \infty$$

and Noga Alon proved that for small c, $h(n; c) < c' n^{\frac{1}{2}}$. It is easy to see and well known that for $c > \frac{1}{4}$ one has $h(n; c) > c_1 n_1$, but without much difficulty the following stronger result can be proved.

Let $e > \frac{1}{4}n^2 - cn$ and assume that every edge of $G(n; e)$ is contained in a triangle. Then there is an absolute constant $c_1 = c_1(c)$ for which our $\mathcal{G}(n; e)$ has an edge which is contained in $\geq c_1 n$ triangles. That the result is best possible is a slight modification of Noga Alon's proof that if $f(n) \to \infty$ then there is a $G(n; (\frac{1}{4}n^2 - nf(n)))$ each edge of which is in a triangle but every edge is only in $o(n)$ triangles.

We give an outline of the proof. Let $G(n; \frac{1}{4}n^2 - cn)$ be a graph each edge of which is contained in a triangle. Observe first that if (x_1, x_2, x_3) is a triangle of

our G, then we can assume that

$$v(x_1)+v(x_2)+v(x_3)\leq n(1+\varepsilon) \tag{1}$$

For if (1) would not hold then one of the edges (x_1, x_2), (x_1, x_3), (x_2, x_3) were contained in at least $\frac{1}{3}(\varepsilon n)$ triangles and thus our theorem is proved in this case. Henceforth we can assume that (1) holds for every triangle of our graph.

Assume next that our $G(n; \frac{1}{4}n^2 - cn)$ contains more than $10c$ vertex disjoint triangles $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$. Omit all these vertices from our G. Then by (1) we obtain a $\mathcal{G}(n-30c)$ which has more that $\frac{1}{4}(n-30c)^2$ edges and therefore by an old and elementary result of mine it contains an edge which is contained in $c'n$ triangles, by the result of Edwards $c' = \frac{1}{6} + o(1)$.

Thus we can assume that (1) holds and our G has at most $10c$ vertex disjoint triangles. But then by a simple argument at least one vertex is contained in $n^2/130c$ triangles and therefore at least one edge is contained in at least $n/130c$ triangles, which proves the first part of our theorem.

The proof of the second half of our assertion is very simple. Let $|G| = n$, $|A| = \ell_n^2$ where ℓ_n tends to infinity as slowly as we please. $|B| = |C| = \frac{1}{2}(n - \ell_n^2)$. Join every vertex of B to every vertex of C. Divide B and C into ℓ_n roughly equal disjoint sets B_i and C_j. Join $x_{i,j} \in A$ to every vertex of B_i and C_j. If ℓ_n tends to infinity sufficiently slowly our graph has $\frac{1}{4}(n^2) - nf(n)$ edges and each edge is on $o(n)$ triangles as stated.

It would perhaps be of interest to improve the estimates for $h(n; c)$ and investigate what happens if $c = c_n \to 0$.

Pyber and I considered the following related problem. Assume again that every edge of a $\mathcal{G}(n; e)$ is contained in at least one triangle. Denote by $L(n; e)$ the largest integer so that our graph contains at least $L(n; e)$ triangles. Trivially for all e $L(n; e) \geq e/3$, and the result of Ruzsa and Szemerédi shows that for $e < cnr_3(n)$ $e/3$ is exact. On the other hand it is well known and easy to see that if $e > (1+\varepsilon)\frac{1}{4}n^2$ then $L(n; e) > c_\varepsilon n^3$ even if we do not assume that every edge is contained in a triangle. We thought that perhaps for $e > cn^2$ $L(n, e) > (\frac{1}{2} + o(1))e$. The complete bipartite graph with a matching shows that if true this is best possible[1]. It would perhaps be of interest to investigate what happens to $L(n; e)$ if $e/n^2 \to 0$ very slowly.

Last year Stephan Burr and I came across the following problem: Let $f(n)$ be the smallest integer for which if we color the edges of $K(f(n))$ (i.e. a complete graph of $f(n)$ vertices) by two colors then there either are two monochromatic $K(n-1)$'s with a common vertex where the two $K(n-1)$'s have different colors or there is a monochromatic $K(n)$. Is it true that $f(n) = r(n; n-1)$? This is open even for $n = 5$.

The following simple problem of Renu Laskar and myself seems still to be

[1] P. Frankl has just proved this conjecture.

open. Let $g(n)$ be the largest integer for which every $\mathcal{G}(n;[\frac{1}{4}n^2]+1)$ contains a triangle x_1, x_2, x_3 for which the sum of the degrees of the vertices x_1, x_2, x_3 is $\geqslant g(n)$. We proved

$$(1+c)n<g(n)<(1+o(1))2(\sqrt{3}-1)n.$$

The upper bound is probably best possible. Clearly many generalisations and extensions are possible.

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CLIQUE PARTITIONS AND CLIQUE COVERINGS

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Several new tools are presented for determining the number of cliques needed to (edge-)partition a graph. For a graph on n vertices, the clique partition number can grow cn^2 times as fast as the clique covering number, where c is at least 1/64. If in a clique on n vertices, the edges between cn^a vertices are deleted, $\frac{1}{2} \leq a < 1$, then the number of cliques needed to partition what is left is asymptotic to c^2n^{2a}; this fills in a gap between results of Wallis for $a < \frac{1}{2}$ and Pullman and Donald for $a = 1$, $c > \frac{1}{2}$. Clique coverings of a clique minus a matching are also investigated.

1. Introduction

Only undirected graphs without loops or multiple edges are considered. The graph K_n on n vertices for which every pair of distinct vertices induces an edge is called a *complete* graph or the *clique* on n vertices. If G is any graph, we call a complete subgraph of G a *clique* of G (we do not require that it be a maximal complete subgraph). A *clique covering* of G is a set of cliques of G which together contain each edge of G at least once; if each edge is covered exactly once we call it a *clique partition*. The *clique covering number* $\mathrm{cc}(G)$ of G is the smallest cardinality of any clique covering; the *clique partition number* $\mathrm{cp}(G)$ is the smallest cardinality of a clique partition.

The question of calculating these numbers was raised in 1977 by Orlin [6]. Already in 1948 deBruijn and Erdős [2] had proved that partitioning K_n into smaller cliques requires at least n cliques. Some more recent calculations motivating this study include [8] by Wallis in 1982, where it is shown that if G has $o(\sqrt{n})$ vertices, then $\mathrm{cp}(K_n - G)$ is asymptotically equal to n; [7] by Pullman and Donald in 1981, where $\mathrm{cp}(K_n - K_m)$ is calculated exactly for $m \geq \frac{1}{2}n$; and in [1] by Cacceta et al. in 1985, where it is found that at its largest $\mathrm{cp}(G) - \mathrm{cc}(G)$ is asymptotic to $\frac{1}{4}n^2$, where G has n vertices.

Several questions left open in these earlier papers are explored. We obtain asymptotic results for $\mathrm{cp}(K_n - K_m)$ for m in the range $\sqrt{n} < m < n$, connecting the results of Wallis 1982 [8] and Pullman 1981 [7]; for example if $m = cn^a$, $\frac{1}{2} < a < 1$, then $\mathrm{cp}(K_n - K_m)$ is asymptotic to c^2n^{2a}. We apply bounds developed in this connection to bound the maximum value of $\mathrm{cp}(G)/\mathrm{cc}(G)$ on graphs G with

n vertices, showing it can grow as fast as cn^2 where $c > \frac{1}{64}$. We also provide simple proofs of some bounds on $cc(T_n)$ where T_n is K_n minus a matching.

2. Lower bound techniques

Let $G = G_n$ be a graph with n vertices, with these vertices divided into two sets A and B with a and b elements, $a + b = n$. The edges of G now fall into three classes which we call "A edges", "B edges", and "connecting edges" depending as their endpoints lie both in A, both in B, or one in each. Suppose a clique in G contains more than one of the connecting edges of G; then it must contain some A edges or B edges or both. If the number of connecting edges in G is large, there will not be enough A edges or B edges of G available to combine the connecting edges into just a few cliques. This technique is used in Theorem 3 of [7], which says (here $C + D$ is the graph that has vertex disjoint copies of graphs C and D and all edges between vertices of C and vertices of D):

Theorem. *Let H be a graph with p vertices and m edges. Let q be at least the edge chromatic number of H. Then $cp(H + \bar{K}_q) = pq - m$ and any minimal clique partition has edges and triangles only.*

The proof depends on the fact that there are pq edges connecting H to $\bar{K}_q$; two lie in the same clique only if that clique contains at least one edge selected from the m edges of H, since $\bar{K}_q$ has no edges.

A similar strategy is used in [1] to produce a sequence of graphs G_n with n vertices for which $cp(G_n) - cc(G_n)$ is asymptotic to $\frac{1}{4}n^2$.

Our goal in this section is to give several lemmas that consider cases where the cliques use more than one connecting edge. We are able to extend several existing lower bounds on clique partition numbers by this strategy.

We begin with a purely numerical lemma.

Lemma 1. *Let $\sum_{i=1}^{q} e_i = c$ and $\sum_{i=1}^{q} e_i^2 \leq d$. Then $q \geq c^2/d$.*

Proof. Substituting the q equal values c/q for the (possibly distinct) e_i preserves the sum of the e_i and can only reduce the sum of the e_i^2. thus $\sum_{i=1}^{q} (c/q)^2 \leq d$ so $q(c/q)^2 \leq d$, and the result follows. □

Lemma 2. *Suppose the graph G has k edges in side A, no edges in side B, and c connecting edges. Then*

$$cp(G) \geq \frac{c^2}{(2k + c)}.$$

Proof. If G is partitioned by q cliques and clique i has e_i connecting edges, then

clique i has $e_i(e_i - 1)/2$ edges in side A. Then $\sum_{i=1}^{q} e_i = c$, and

$$k \geq \sum_{i=1}^{q} e_i(e_i - 1)/2 = \left(\sum_{i=1}^{q} e_i^2 - \sum_{i=1}^{q} e_i\right)\Big/2 = \left(\sum_{i=1}^{q} e_i^2 - c\right)\Big/2$$

and the result follows from Lemma 1. □

We thus obtain a lower bound for the clique partition number of a clique minus a clique:

Theorem 1. *From $n \geq m \geq 1$, and $n \neq 1$,*

$$cp(K_n - K_m) \geq \frac{(n-m)m^2}{(n-1)}.$$

Proof. There are $n - m$ points and $\frac{1}{2}(n-m)(n-m-1)$ edges in "side A" and $m(n-m)$ connecting edges; Lemma 2 applies. □

Corollary 1. *If $0 < c < 1$, then*

$$cp(K_n - K_{cn}) \geq (n - cn)(cn)^2/(n-1) \approx (1-c)c^2n^2.$$

Here $f \approx g$ means that as $n \to \infty$, $f/g \to 1$.

In [7] there is a corollary of Theorem 3 which gives an exact formula: $\text{cp}(K_n - K_m) = \frac{1}{2}(n-m)(3m-n+1)$ when $n > m \geq \frac{1}{2}(n-e)$ (where $e = 0$ for $n - m$ odd, $e = 1$ otherwise). Our result is not as good as theirs for $m > \frac{1}{2}n$ (for example, for $n = 12$, $m = 8$, $c = \frac{2}{3}$, we get $\text{cp}(G) > 23$ and they get $\text{cp}(G) = 26$) but our result gives some indication of the value of $\text{cp}(K_n - K_{cn})$ even if $m = cn$ is a small fraction of n.

Wallis has told us that Rose, a student of Pullman, has obtained exact results for $m < \frac{1}{2}n$; we have not seen these independent results.

In a sense, our result fails to be tight for two reasons:

(1) there may be cliques using no connecting edges, if m is small.

(2) Lemma 1 uses an averaging process: in actual practice no clique can have a fractional number of edges, so the e_j are not normally equal in a minimal partition.

Corollary 2. *If $\frac{1}{2} < a < 1$ and $m = cn^a$, then for n large enough $cp(K_n - K_m) \geq (n - cn^a)c^2n^{2a}/(n-1) \approx c^2n^{2a}$.*

This result will be discussed further once the corresponding upper bound is found, in Section 4.

We now turn to results that apply if there are edges in both 'sides' of G. The first pair of lemmas are useful when the cliques involved are typically very small.

Lemma 3. *Let a clique K_r have u edges in side A, v edges in side B, and s connecting edges. Then*

$$s-1 \leqslant u+v+\min(u, v).$$

Proof. Suppose that A and B have a and b vertices as usual; suppose for concreteness that $a \leqslant b$. Now $s=ab$, $u=a(a-1)/2$, and $v=b(b-1)/2$. If $a=b$, then clearly (defining $P(u, v)$)

$$\begin{aligned} P(u, v) &= u+v+\min(u, v)+1-s \\ &= 3a(a-1)/2+1-a^2 \\ &= (a-1)(a-2)/2 \geqslant 0 \end{aligned}$$

Now whenever b grows by 1, $p(u, v)$ grows by $b-a \geqslant 0$, so $p(u, v) \geqslant 0$ as required. □

Aggregating a number of such cliques partitioning a graph G, we obtain:

Lemma 4. *Let G have u edges in side A, v edges in side B, and s connecting edges. Then*

$$cp(G) \geqslant s-u-v-\min(u, v).$$

Proof. Suppose G is covered by q cliques with the number of edges in the parts of cliques i being u_i, v_i, and s_i. Thus Lemma 3 implies $1 \geqslant s_i-u_i-v_i-\min(u_i, v_i)$, and summing for $i=1, \ldots, q$, we have

$$\begin{aligned} q=\sum_{i=1}^{q} 1 &\geqslant \sum_{i=1}^{q} s_i-\sum_{i=1}^{q} u_i-\sum_{i=1}^{q} \min(u_i, v_i) \\ &= s-u-v-\sum_{i=1}^{q} \min(u_i, v_i) \\ &\geqslant s-u-v-\min\left(\sum_{i=1}^{q} u_i, \sum_{i=1}^{q} v_i\right) \\ &= s-u-v-\min(u, v). \end{aligned}$$

Example 1. Consider the graph G_n defined as follows: $\frac{1}{2}n$ vertices are in a clique A; the other $\frac{1}{2}n$ vertices forming set B are divided into 4 cliques each of $\frac{1}{8}n$ vertices; all the vertices of A are connected to all the vertices of B. Then, there are $\frac{1}{2}(\frac{1}{2}n)(\frac{1}{2}n-1)$ edges in side A, $2(\frac{1}{8}n)(\frac{1}{8}n-1)$ edges in side B, and $\frac{1}{4}n^2$ connecting edges. By Lemma 4,

$$\begin{aligned} \mathrm{cp}(G_n) &\geqslant \tfrac{1}{4}n^2-(\tfrac{1}{8}n-\tfrac{1}{4}n)-2(\tfrac{1}{32}n^2-\tfrac{1}{4}n) \\ &= \tfrac{1}{16}n^2+\tfrac{1}{4}(3n). \end{aligned}$$

Since $\mathrm{cc}(G_n) = 4$ (each of the 4 cliques covers A and one clique of B, a total of $\frac{1}{2}n + \frac{1}{4}n$ vertices), we conclude that

$$\mathrm{cp}(G_n)/\mathrm{cc}(G_n) > \tfrac{1}{64}n^2.$$

It follows that $\mathrm{cp}(G_n)/\mathrm{cc}(G_n)$ can exceed cn^2 where c can be at least $\frac{1}{64}$.

Lemma 3 is wasteful when the number of clique covering connecting edges is large (it is exact only for K_2, K_3, and for K_4 and K_5 when they have exactly two vertices on one side). Here is another approach useful when one or both sides of some connecting cliques are moderately large. Lemma 5 says that goodsized cliques (those with at least m vertices on the larger side) use up 'side' edges at least $(m-1)/m$ times as fast as 'connecting' edges.

Lemma 5. *Let a clique K_r have its vertices partitioned into sets A and B of sizes a and b, $a + b = r$. Suppose $a \geqslant m$. Then*

$$\left(\frac{(m-1)}{m}\right)ab \leqslant \frac{a(a-1)}{2} + \frac{b(b-1)}{2}.$$

Proof. It is easy to check that

$$\begin{aligned} P(a, b) &= \frac{a(a-1)}{2} + \frac{b(b-1)}{2} - \left(\frac{(m-1)}{m}\right)ab \\ &= \left(\frac{1}{m}\right)ab + \frac{(a-b)^2}{2} - \frac{(a+b)}{2} \end{aligned}$$

so we need only check that $P(a, b) \geqslant 0$.

If $a \geqslant m$ and $b \geqslant m$, $(1/m)ab \geqslant \max(a, b) \geqslant \frac{1}{2}(a+b)$ and $P(a, b) \geqslant 0$. For $a = m$ and $0 \leqslant b \leqslant m-1$, $P(a, b)$ is a decreasing function of b and zero only at $b = m-1$, so $P(m, b) \geqslant 0$. Fixing $b \leqslant m-1$ and supposing $a \geqslant m$, $P(a, b)$ is an increasing function of a, completing the proof. □

The use of Lemma 5 is somewhat tricky; it is included primarily because it allows us to cope with the following example.

Example 2. Dom de Caen asked (question communicated to us orally by Pullman and by Wallis) about the clique partition number of the graph G_{3n} composed of three copies of K_n with all vertices in the second copy joined to all vertices in the first and third (in our notation, loosely, $K_n + K_n + K_n$). In particular, does it grow proportionally to n^2? We can prove that it does.

Treating the second K_n as side A and the other two as side B, A has $n(n-1)/2$ edges and B has $n(n-1)$ edges, with $n(2n)$ connecting edges. Suppose there are q cliques in a clique partition, where the ith clique has a_i vertices in side A and b_i vertices in side B (hence all b_i vertices lie in the first K_n or all lie in the third K_n;

no edge connects those two cliques). Suppose the q cliques are so ordered that $(a_1, b_1), \ldots, (a_r, b_r)$ all have $a_i < m$ and $b_i < m$, while $(a_{r+1}, b_{r+1}), \ldots, (a_q, b_q)$ all have $a_i \geq m$ and/or $b_i \geq m$. Now for $j = 1, \ldots, r$ we have

$$\frac{a_j(a_j - 1)}{2} + \frac{b_j(b_j - 1)}{2} \geq 0 \geq \left(\frac{(m-1)}{m}\right)(a_j b_j - (m-1)^2)$$

while for $j = r + 1, \ldots, q$ we have

$$\frac{a_j(a_j - 1)}{2} + \frac{b_j(b_j - 1)}{2} \geq \left(\frac{(m-1)}{m}\right) a_j b_j.$$

Summing over all q cliques,

$$\sum_{i=1}^{q} \frac{a_j(a_j - 1)}{2} + \sum_{i=1}^{q} \frac{b_j(b_j - 1)}{2} \geq \left(\frac{(m-1)}{m}\right) \sum_{i=1}^{q} a_j b_j - \frac{q(m-1)^3}{m}.$$

But

$$\sum_{i=1}^{q} \frac{a_j(a_j - 1)}{2} = \frac{n(n-1)}{2},$$

and

$$\sum_{i=1}^{q} \frac{b_j(b_j - 1)}{2} = n(n-1),$$

so

$$\frac{3n(n-1)}{2} \geq \left(\frac{(m-1)}{m}\right)(2n^2) - \left(\frac{(m-1)^3}{m}\right) q$$

and

$$q \geq \left(\frac{(m-4)}{(m-1)^2}\right) n^2.$$

The right hand side of the above inequality, considered as a function of m, has a maximum when $m = 6$ (m must be an integer). Therefore, $q > 2n^2/125$.

Thus de Caen's conjecture that this graph has a fast-growing clique partition number is correct. However, our methods do not establish a large enough value of $\text{cp}(G_{3n})$ to suggest that $\text{cp}(G_{3n})/\text{cc}(G_{3n})$ grows as fast as in Example 1. Of course, in neither case have we established an exact value for $\text{cp}(G)$; we have only a lower bound.

3. Upper bounds for a clique minus a clique and cp/cc

Here we modify a strategy used in [8] to provide an upper bound for some of the clique partition numbers bounded below in Section 2.

Theorem 2. *If $m = f(n)$ and for large enough n, $\sqrt{n} < m < n$, then $cp(K_n - K_m) < m^2 + o(m^2)$.*

Proof. Let p be a prime power at most slightly larger than m; there are constants $0<c<1$ and $0<b<1$ so that for large enough m, there is a p with $m<p<m+cm^b$. Then $p^2=m^2+o(m^2)$. In a projective plane of parameter p, delete a line of $p+1$ points leaving p^2 points in p "parallel" lines. In one of those lines, delete all but m points; in the other lines, delete a total of $(p^2-n)-(p-m)$ points. This leaves a total of n points, with m of them on a selected line. Use this design to construct a clique partition of K_n-K_m into at most p^2+p-1 cliques: each line is a clique, except the selected line of m points. There are $p-1$ other "parallel" lines and p^2 "crossing" lines. Hence

$$\mathrm{cp}(K_n-K_m)\leq p^2+p-1\leq m^2+o(m^2)$$

as desired. □

Corollary 1. *If $m=cn^a$ with $\frac{1}{2}<a<1$, then $cp(K_n-K_m)$ is asymptotic to c^2n^{2a}.*

Proof. An upper bound is given by Theorem 2 and a lower bound by Corollary 2 of Theorem 1. □

This result fills in most of the gap between the results of Wallis (if $m\leq\sqrt{n}$, $\mathrm{cp}(K_n-K_m)\approx n$) and of Pullman and Donald (if $m=\frac{1}{2}n$, $\mathrm{cp}(K_n-K_m)=\frac{1}{2}(\frac{1}{2}n)(\frac{1}{2}n+1)\approx\frac{1}{2}(\frac{1}{2}n)^2$). There is still a gap: our result is poorer than that of Pullman and Donald by a factor of two.

More generally, we do not get as clean a result as Corollary 1 for the case $m=cn$; Corollary 1 to Theorem 1 gives a lower bound of $(1-c)c^2n^2$ while Theorem 2 yields an upper bound of c^2n^2.

We now turn to providing an upper bound for $\mathrm{cp}(G)/\mathrm{cc}(G)$. We are able to do little here other than some delineation of the problem. We have seen in Section 2 that if G has n vertices we can have $\mathrm{cp}(G)/\mathrm{cc}(G)>\frac{1}{64}n^2$. How big can it get? It is already known from [3] that $1\leq\mathrm{cc}(G)\leq\mathrm{cp}(G)\leq\frac{1}{4}n^2$. If $\mathrm{cc}(G)=1$, then also $\mathrm{cp}(G)=1$ so we need consider only cases with $2\leq\mathrm{cc}(G)$. Hence, we already see that $\mathrm{cp}(G)/\mathrm{cc}(G)\leq\frac{1}{8}n^2$. The following proposition improves this result slightly for large enough n.

Proposition 1. *If G has n vertices, and n is large enough $cp(G)/cc(G)\leq\frac{1}{12}n^2$.*

Proof. If $\mathrm{cc}(G)\geq 3$, we are done. Thus we can suppose $\mathrm{cc}(G)=2$ and must show that $\mathrm{cp}(G)\leq\frac{1}{6}n^2$. In fact, we do better: we obtain $\mathrm{cp}(G)\leq\frac{1}{8}n^2$.

Since $\mathrm{cc}(G)=2$, G can be covered by two cliques K_a and K_b intersecting in a clique K_c; $a+b-c=n$. Suppose for concreteness that $a\leq b$. If $c\leq\frac{1}{3}n$, cover K_b with 1 clique and partition $G-K_b=K_a-K_c$ with $c^2+o(c^2)$ cliques. Since $c^2\leq\frac{1}{9}n^2$, for n large enough $\mathrm{cp}(G)<\frac{1}{8}n^2$ as desired.

If $c>\frac{1}{3}n$, then let $c=\frac{1}{3}n+x$ for $x>0$. Since $a\leq b$, $a-c\leq\frac{1}{2}(n-c)\leq\frac{1}{3}n-\frac{1}{2}x$,

and therefore $c > \frac{1}{2}a$. By [7] $K_a - K_c$ can be partitioned by exactly $\frac{1}{2}(a-c)(3c-a+1)$ cliques. Also, there are $(a-c)c$ edges between K_c and K_{a-c} (the clique with vertices in K_a not in K_c). We will consider two clique partitions of G: (1) a clique partition of $K_a - K_c$ along with the clique K_b, and (2) the edges between K_c and K_{a-c} along with the clique K_b and K_{a-c}. If either of these partitions has at most $\frac{1}{8}n^2$ elements, the proof is complete. Thus, we need one of the following inequalities to hold.

$$\frac{\left(\frac{n}{3}-\frac{x}{2}\right)\left(\frac{n}{3}+\frac{5x}{2}+1\right)}{2}+1 \leq \frac{n^2}{8} \tag{1}$$

$$\left(\frac{n}{3}-\frac{x}{2}\right)\left(\frac{n}{3}+x\right)+2 \leq \frac{n^2}{8} \tag{2}$$

The inequality (2) is satisfied for $|x - \frac{1}{6}n| \leq 2$, and the inequality (1) is satisfied for the remaining values of x. This completes the proof of the proposition. □

4. A clique minus a matching

In [6] Orlin defines T_{2n} (not a tree) to be the graph obtained by deleting a perfect matching (a set of n edges, no vertex on two of them) from K_{2n}. He asks about clique coverings and clique partitions of T_{2n}. In [5] Gregory and Pullman establish that $\mathrm{cc}(T_{2n}) \approx \log n$; in [4] Gregory et al. show that $\mathrm{cp}(T_n) \geq n$ for $n \geq 8$ and that asymptotically, $\mathrm{cp}(T_n) \leq n \log\log n$. The last upper bound is proved by methods strikingly similar to those in the previous section.

We here offer bounds on $\mathrm{cc}(T_n)$ obtained by methods motivated by the heuristic discussion in [6]. These results are less precise than those in [5], but may be easier to visualize.

In order to discuss clique coverings of T_n, we need some notation for the vertices. Suppose $m = \frac{1}{2}n$; T_n will be considered to have vertices a_i and b_i for $i = 1, \ldots, m$. All edges are present except the edges from a_i to b_i for $i = 1, \ldots, m$. Note that no clique in a covering can contain an a_i and the corresponding b_i but that there must be cliques containing each a_i, b_j pair with $i \neq j$ as well as ones containing each a_i, a_j pair and each b_i, b_j pair.

Theorem 3. *For all n, $cc(T_n) \geq (\log n) - 1$.*

Proof. Clearly T_n cannot be covered by one clique. Given $i \neq j$, there must be a clique in the covering containing a_i and b_j; that clique cannot also contain a_j. Hence, for each $i \neq j$, there is a clique containing a_i but not a_j. But it thus follows easily that there are at least $\log(\frac{1}{2}n)$ cliques. (There is a clique containing a_1 but not a_2. Since there are $\frac{1}{2}n$ a_i's, this clique either includes at least $\frac{1}{4}n$ a_i's or excludes at least $\frac{1}{4}n$ a_i's. Choose the larger such set—the included or excluded a_i's—and find a clique separating two of them. Continue $\log(\frac{1}{2}n)$ times). □

Theorem 4. *For all n, $cc(T_n) \leq 2(\log n)$.*

Proof. We construct an explicit clique covering. For each a_i write our the subscript i as a binary integer of $\log(\frac{1}{2}n)$ digits (e.g. 0001, 0010, 0011, 0100, . . .). If $\frac{1}{2}n$ is a power of two, code the last i as 0000 to avoid the need for an extra digit. Clique A_k will include all a_i for which the kth digit of i is a 1, and all b_i for which the kth digit of i is 0. Clique B_k has the vertices not in A_k. Since if $i \neq j$, i and j differ in at least one binary digit, the edge from a_i to b_j is in at least one A_k or B_k. Finally, add a clique containing all the a_i and a clique containing all the b_i; this yields a complete clique covering having at most $2(\log(\frac{1}{2}n)) + 2$ cliques. □

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EXTREMAL THEORY AND BIPARTITE GRAPH-TREE RAMSEY NUMBERS

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For a positive integer n and graph B, $f_B(n)$ is the least integer m such that any graph of order n and minimal degree m has a copy of B. It will be show that if B is a bipartite graph with parts of order k and l ($k \leq l$), then there exists a positive constant c, such that for any tree T_n of order n and for any j ($0 \leq j \leq (k-1)$), the Ramsey number

$$r(T_n, B) \leq n + c \cdot (f_B(n))^{j/(k-1)}$$

if $\Delta(T_n) \leq (n/(k-j-1)) - (j+2) \cdot f_B(n)$. In particular, this implies $r(T_n, B)$ is bounded above by $n + o(n)$ for any tree T_n (since $f_B(n) = o(n)$ when B is a bipartite graph), and by $n + O(1)$ if the tree T_n has no vertex of large degree. For special classes of bipartite graphs, such as even cycles, sharper bounds will be proved along with examples demonstrating their sharpness. Also, applications of this to the determination of Ramsey number for arbitrary graphs and trees will be discussed.

1. Introduction

For graphs G and H, the *Ramsey number* $r(G, H)$ is the least integer N such that in any two-coloring (say with colors red and blue) of the edges of K_N, there is either a copy of G in the red subgraph or a copy of H in the blue subgraph. We investigate the Ramsey number $r(T_n, B)$, where T_n denotes a tree on n vertices and B is a bipartite graph.

Let B be a bipartite graph with parts of order k and l ($k \leq l$). Thus $B \subseteq K_{k,l}$, the complete bipartite graph. For any positive integer n, let $f_B(n)$ be the smallest positive integer m such that any graph of order n and minimal degree m contains a copy of B. The *extremal degree number* $f_B(n)$ is related to the *extremal number* $\text{ext}_B(n)$, which is the minimum number of edges in a graph of order n which insures that there is a copy of B. In fact, $\text{ext}_B(n) \geq n \cdot f_B(n)/2$ with the two expressions essentially the same for many graphs B. Therefore, $f_B(n) = o(n)$ for any bipartite graph, in fact, $f_B(n) \leq c \cdot n^{(k-1)/k}$ for an appropriate constant c [13].

The main result that will be proved is the following, which gives an upper bound for the Ramsey number $r(T_n, B)$.

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Theorem 1. *For a fixed bipartite graph $B \subseteq K_{k,l}$ $(k \leq l)$ there exists a positive constant c such that for any j* $(0 \leq j \leq k-1)$, *and any tree* T_n,

$$r(T_n, B) \leq n + c \cdot (f_B(n))^{j/(k-1)},$$

when $\Delta(T_n) \leq (n/(k-j-1)) - (j+2) \cdot f_B(n)$.

For $k=2$ or 3, the bounds given in Theorem 1 are of the right order of magnitude, and cannot be improved. In [9] it is proved that if $m = \Delta(T_n)$, then

$$r(T_n, C_4) = \max\{4, n+1, r(K_{1,m}, C_4)\}.$$

Also, $r(K_{1,m}, C_4) \leq m + c' \cdot m^{\frac{1}{2}}$, which is consistent with the degree extremal number for C_4. This verifies the sharpness of Theorem 1 for $k = l = 2$. For $k=2$ or 3 and l arbitrary, there are similar results in [9] indicating the sharpness of Theorem 1. For $k \geq 4$, little is known about the extremal numbers of $K_{k,l}$, so it is difficult to measure how accurate the results of Theorem 1 are.

The two extreme cases of Theorem 1 ($j = k-1$ and $j = 0$) give the following two corollaries. When $j = k-1$, there is no restriction on the degree of vertices in T_n.

Corollary 2. *For a fixed bipartite graph B, there is a positive constant c such that for all trees* T_n *of order n*,

$$r(T_n, B) \leq n + c \cdot f_B(n).$$

The above corollary implies that for any tree T_n and bipartite graph B, $r(T_n, B) = n + o(n)$. For special classes of trees, such as those with no vertices of large degree, $r(T_n, B) = n + O(1)$. This follows from the next corollary.

Corollary 3. *For a fixed bipartite graph $B \subseteq K_{k,l}$ $(k \leq l)$ there exists a positive constant c such that for any tree* T_n,

$$r(T_n, B) \leq n + c,$$

when $\Delta(T_n) \leq (n/(k-1)) - 2 \cdot f_B(n)$.

When $B = C_4$, the constant c in Corollary 3 was shown to be 1 in [9]. It is conjectured that in fact $c = k-1$ will suffice in the general case. It is, of course, impossible to find a better constant than this, since $K_{k-1,n-1}$ contains no $K_{k,l}$ and its complement contains no connected graph of order n.

The techniques used to prove Theorem 1 can be used to obtain sharper bounds for special classes of bipartite graphs such as even cycles. Corollary 2 implies that

$$r(T_n, C_{2k}) \leq n + c \cdot n^{1/k},$$

since $f_B(n) \leq a \cdot n^{1/k}$ for $B = C_{2k}$ [2]. The next result gives an improvement of this bound when there are no vertices of extremely large degree.

Theorem 4. *For any integer $k \geq 2$, there exists positive constants c and d such that*

$$r(T_n, C_{2k}) \leq n + c$$

for any tree T_n of order n with $\Delta(T_n) \leq n - d \cdot n^{1/k}$.

2. Notation and terminology

Notation will generally follow that used in [1]. However, some special conventions will be used. We describe some of the special and most often used terminology.

By a *two-coloring* of a complete graph K_N we will always mean a coloring of the edges of K_N using red (R) for the first color and blue (B) for the second color. The red subgraph will be denoted by $\langle R \rangle$ and the blue subgraph by $\langle B \rangle$.

By T_m we will mean a tree of order m. A path in a graph G in which all of the interior vertices have degree two in G is called a *suspended path*. An *end-vertex* is a vertex of degree 1, and an *end-edge* is an edge incident to an end-vertex. End-edges are *independent* if no pair of them is incident. A *talon of degree m* consists of a vertex incident to m end-edges of the graph.

A bipartite graph B with parts of order k and l will be denoted by $B_{k,l}$. Thus, $B_{k,l} \subseteq K_{k,l}$. The minimum degree and maximum degree of vertices of a graph G will be denoted by $\delta(G)$ and $\Delta(G)$ respectively. The *neighborhood* of a vertex v of G will be denoted by $N_G(v)$, and the neighborhood of a set S of vertices (which is the union of the neighborhoods of the vertices of S) will be denoted by $N_G(S)$. If H is a subgraph of G, then $G - H$ is the graph obtained from G by deleting the vertices of H and any incident edges.

3. Proofs

Before proving Theorem 1 and Theorem 4, we will prove some lemmas that will handle special cases, and state some known results that will be helpful. A basis for the proof is that any large tree will have either a long suspended path, many independent end-edges, or a large degree talon. The first lemma deals with trees with long suspended paths, and the second lemma with trees with many independent end-edges.

Lemma 5. *For $l \geq k$ and n positive integers, let T_{n-1} be a tree with a suspended path of $l(k+l)$ vertices and T_n the tree obtained from T_{n-1} by subdividing one edge on the suspended path. If a K_{n+k-1} is two-colored such that $T_{n-1} \subseteq \langle R \rangle$, then either $T_n \subseteq \langle R \rangle$ or $B_{k,k} \subseteq \langle B \rangle$.*

Proof. Let $X = (x_1, x_2, \ldots, x_m)$ for $m = l(k+l)$ be the suspended path of T_{n-1} in $\langle R \rangle$ and let Y be the k vertices of K_{n+k-1} not in the T_{n-1}.

Assume that $T_n \not\subseteq \langle R \rangle$. Then no vertex of Y is adjacent in $\langle R \rangle$ to two consecutive vertices of X. Also, if $x_i y, x_j y \in \langle R \rangle$ for $y \in Y$, then (assuming x_i and x_j have successors along X) $x_{i+1}y, x_{j+1}y$ and $x_{i+1}x_{j+1} \in \langle B \rangle$. Therefore, if a vertex of Y is adjacent to $k+l$ vertices of X in $\langle R \rangle$, then $\langle B \rangle \supseteq K_{k+l} \supseteq K_{k,l}$. Thus, we can assume that each vertex of Y is adjacent in $\langle R \rangle$ to at most $k+l-1$ vertices of X. This implies that at least l vertices of X are adjacent to each vertex of Y in $\langle B \rangle$, which completes the proof. □

Lemma 6. *For $n > m \geq k$ and l positive integers, let T_n be a tree obtained from a tree T_{n-m} by adding m independent end-edges. Then, $r(T_n, B_{k,l}) \leq \max\{r(T_{n-m}, B_{k,l}) + kl^2, n+k-1\}$.*

Proof. Let $r = \max\{r(T_{n-m}, B_{k,l}) + kl^2, n+k-1\}$ and consider a two-coloring of the graph $G = K_r$ such that $\langle B \rangle \not\supseteq B_{k,l}$, and $\langle R \rangle \not\supseteq T_n$. We will show that this leads to a contradiction.

Successively select vertex disjoint subgraphs $H_1, H_2, \ldots, H_l$ in $\langle B \rangle$ as follows: H_i is disjoint from $H_1, \ldots, H_{i-1}$ and contains a maximal number of vertices while still being isomorphic to a subgraph of $K_{k,l}$. Since $\langle B \rangle \not\supseteq K_{k,l}$, each vertex not in H_i is adjacent in $\langle R \rangle$ to at least one vertex of H_i. Let H be the union of these subgraphs. Therefore each vertex of $G - H$ is adjacent in $\langle R \rangle$ to at least one vertex in each H_i ($1 \leq i \leq l$). By assumption there is an embedding τ of T_{n-m} into $\langle R \rangle$ such that $\tau(T_{n-m})$ is disjoint from H.

Let X be the m vertices of T_{n-m} incident to the m independent end-edges of T_n not in T_{n-m}, and let Y be the vertices of G not in $\tau(T_{n-m})$. Consider the bipartite graph L with parts $\tau(X)$ and Y induced by $\langle R \rangle$. A matching in the graph L which saturates $\tau(X)$ would imply that $\langle R \rangle \supseteq T_n$, so assume no such matching exists. Therefore, by Hall's theorem [12], there is a subset S of $\tau(X)$ such that $|N_L(S)| < |S|$. Since $V(H) \subseteq Y$, each vertex of $\tau(X)$ has degree at least l and $|S| > l$. Therefore, the vertices of S are commonly adjacent in $\langle B \rangle$ to at least $|Y| - m + 1 \geq k$ vertices of Y. This gives a $K_{k,l}$ in $\langle B \rangle$, a contradiction which completes the proof. □

The following lemma is used to verify that a tree without long suspended paths and many independent end-edges must have a large talon.

Lemma 7 [4]. *If a tree T_n does not contain any suspended path with more than s vertices, then the number of end-vertices of T_n is at least $n/(2s)$.*

The next lemma is a technical result about the extremal degree number. It is intuitively obvious and convenient for some calculations in the proof of Theorem 1.

Lemma 8. *Let $B = B_{k,l}$ and $N = n + c \cdot (f_B(n)^\alpha)$ for a constant $c > 0$ and $0 < \alpha \leq 1$. Then, for large n, $f_B(N) < 2 \cdot f_B(n)$.*

Proof. Clearly, $f_B(N) \leq f_B(n) + c \cdot (f_B(n))^\alpha$. If $\alpha < 1$, the result follows immediately. The same is true if B is a forest, for then $f_B(n)$ is bounded. Thus, we assume that $\alpha = 1$, and B contains an even cycle, so $f_B(n)$ is unbounded [13].

Let G be a graph of order N with $\delta(G) \geq 2 \cdot f_B(n)$, H a subgraph of order n, and $S = V(G - H)$. We will assume that $H \not\supseteq B$, and show that this leads to a contradiction. Since $H \not\supseteq B$, $\delta(H) < f_B(n)$ and there is an $h_1 \in H$ which is adjacent to at least $f_B(n)$ vertices of S. Assume that $h_1, \ldots, h_i$ have been shown, and let $H_i = H - \{h_1, \ldots, h_i\}$. Again, $\delta(H_i) < f_B(n)$, so there exists an $h_{i+1} \in H_i$ adjacent to at least $f_B(n) - i$ vertices of S. For $m = \lceil f_B(n) \rceil$ and $H' = \{h_1, \ldots, h_m\}$, each vertex of H' is adjacent to m vertices of S.

There are at least $\binom{m}{k}$ k-subsets of S in the neighborhood of each of the m vertices of H'. However, there are only $\binom{cm}{k}$ k-subsets of S. Thus, for m large, some k-subset is in the neighborhood of at least $m \cdot \binom{m}{k} / \binom{cm}{k} \geq l$ vertices of H'. This implies $G \supseteq B$, a contradiction which completes the proof. □

The major difficulty in proving both Theorem 1 and Theorem 4 is dealing with the case of trees with large talons. The following is a greedy algorithm that will be used in embedding such trees.

Algorithm. Our objective is to describe a procedure to assist in embedding a tree T_n in $\langle R \rangle$ of a two-colored $G = K_{n+t}$ in which $\langle B \rangle \not\supseteq K_{k,l}$, and $t \geq l$. We will assume that the tree T_n contains a talon with q edges and that $\delta(\langle R \rangle) \geq n - q$.

Let v be the center of the talon, and denote the tree obtained from T_n by deleting the q edges of the talon by T_{n-q}. Let w be a vertex of maximal degree in $\langle R \rangle$, and S the vertices adjacent in $\langle B \rangle$ to w. Clearly T_{n-q} can be embedded in $\langle R \rangle$, since $\delta(\langle R \rangle) \geq n - q$, but our objective is to do this embedding in such a way that the end-vertices of the talon can also be embedded. To achieve this, we would like to use as many vertices of S as possible when we embed T_{n-q}.

Define the embedding τ of T_{n-q} as follows:

(1) Root the tree T_{n-q} at v and set $\tau(v) = w$.

(2) For $u \in V(T_{n-q})$, suppose that $\tau(u)$ has been defined, and $u_1, u_2, \ldots, u_m$ are the children of u. Select the images $\tau(u_1), \ldots, \tau(u_m)$ such that the edges $\tau(u)\tau(u_1), \ldots, \tau(u)\tau(u_m) \in \langle R \rangle$, and such that a maximum number of the vertices of $\{\tau(u_1), \ldots, \tau(u_m)\}$ are in S. If all of these vertices are not in S, label the vertex u "bad" and place it in the set D. The vertex v will always be considered a "bad" vertex.

This defines an embedding τ of T_{n-q} into $\langle R \rangle$, since $\delta(\langle R \rangle) \geq n - q$. Let S' be the vertices of S not in the image of τ. Three situations can occur.

(a) If $|S'| \leq t$, then the embedding τ can be extended to T_n, since there will be at least q vertices adjacent to w in $\langle R \rangle$ which are not in $\tau(T_{n-q})$.

(b) If $|S'| > t$, and the number of "bad" vertices $|D| \geq k$, then $\langle B \rangle \supseteq K_{k,l}$, since all edges between D and S' are in $\langle B \rangle$. This cannot occur.

(c) If $|S'| > t$, and the number of "bad" vertices $|D| < k$, then many edges of T_{n-q} will be incident to these "bad" vertices. In fact, each edge of T_{n-q} is

either embedded in S, or is incident to a "bad" vertex. Therefore, at least $(n-|S-S'|)$ vertices of T_{n-q} are adjacent to the "bad" vertices. When $|S|$ is small in comparison to n, this will be used to generate vertices of large degree in T_n. In fact, T_{n-q} must contain a vertex of degree at least $(n-|S-S'|)/|D|$. This will give a contradiction under appropriate conditions that will exist when the algorithm is applied.

Proof of Theorem 1. The proof will be by induction on n, the order of the tree. An appropriate choice of c insures that the result is true for small values of n. Assume the theorem is true for all trees of order less than n, and that n is large. Let $M(j)=\lfloor c\cdot(f_B(n))^{j/(k-1)}\rfloor$ and $N=n+M(j)$, and assume that $G=K_N$ is two-colored such that $\langle R\rangle \not\supseteq T_n$ and $\langle B\rangle \not\supseteq B$. We will show that this leads to a contradiction.

The remainder of the proof will be broken into three cases:
(1) T_n has a suspended path with at least $l(k+l)+1$ vertices
(2) T_n has kl^2 independent end-edges, or
(3) T_n has a talon with at least $n/(2kl^3(k+l))$ edges.

These cases are exhaustive. If (1) does not occur, then T_n has a least $n/(2l(k+l))$ end-edges by Lemma 7. If (2) does not occur, then all these end-edges are involved in at most kl^2 talons, which gives (3).

Case (1). T_n has a suspended path with at least $l(k+l)+1$ vertices

Let T_{n-1} denote the tree obtained from T_n by decreasing the length of the suspended path by 1. By the induction assumption, $\langle R\rangle \supseteq T_{n-1}$. An appropriate choice of the constant c insures that Lemma 5 applies, which gives a contradiction in this case.

Case (2). T_n has kl^2 independent end edges

Let $m=kl^2$, and let T_{n-m} be the tree obtained from T_n by deleting m independent end-edges. Lemma 6 implies

$$\begin{aligned} r(T_n,B) &\le \max\{n-m+c\cdot(f_B(n-m))^{j/(k-1)}+kl^2,\ n+k-1\} \\ &\le n+c(f_B(n))^{j/(k-1)} \end{aligned}$$

for appropriate choice of c. This contradiction completes the proof of this case.

Before considering Case (3), we will make some general observations about $\langle R\rangle$ and the degree of vertices in this subgraph. Note that by the definition of $f_B(n)$, $\Delta(\langle R\rangle)\ge N-f_B(N)$. By Lemma 8, $f_B(N)\le 2f_B(n)$, so $\Delta(\langle R\rangle)\ge N-2\cdot f_B(n)$. Also, the number of vertices of "small" degree in $\langle R\rangle$ is small. Consider any number p $(0<p<1)$, and let x be the number of vertices of $\langle R\rangle$ of degree less than $(1-p)n$. Each of these vertices has degree at least $\lceil pn\rceil$ in $\langle B\rangle$

and at least $\binom{\lceil pn \rceil}{k}$ subsets of cardinality k in its neighborhood. Since $\langle B \rangle \not\supseteq B$,

$$x \cdot \binom{\lceil pn \rceil}{k} \leq (l-1)\binom{N}{k}.$$

This implies that x is bounded by a function that depends only on k, l, and p, and not on n. These x vertices can be deleted without significantly changing either the number or degree of the remaining vertices (appropriately alter the constants c and p). Thus throughout the remainder of the proof we will assume that $\delta(\langle R \rangle) \geq (1-p)n$. The appropriate choice for the value of p will depend on the conditions in Case (3), which follows.

Case (3). T_n has a talon with at least $n/(2kl^3(k+l))$ edges.

Select p $(0<p<1)$ such that pn is the maximal degree of a talon in T_n. Thus certainly $pn \geq n/(2kl^3(k+l))$. Let v be the center of this talon, and T_{n-q} the tree obtained from T_n by deleting the q edges of the talon, where $q=pn$. Also, let w be a vertex of maximal degree in $\langle R \rangle$, and S the vertices adjacent to w in $\langle B \rangle$. Since $\Delta(\langle R \rangle) \geq N - 2 \cdot f_B(n)$, S has at most $2 \cdot f_B(n)$ vertices. We apply the algorithm described earlier (with $t = M(j)$). Notation used in the description of the algorithm will be used in the following discussion.

Three subcases $j=k-1$, $j=0$, and $1 \leq j < k-1$ will be considered.

$j=k-1$

If $c \geq 2$, the algorithm yields an embedding, since $|S|$, and hence $|S'|$, is less than t and (a) of the algorithm applies. This gives a contradiction.

$j=0$

In this case we can assume that neither (a) or (b) of the algorithm applies for otherwise we would have a contradiction. Therefore, there are at most $k-1$ "bad" vertices, and one of these vertices has degree at least $(n-|S|)/(k-1)$ by (c), which contradicts the condition on $\Delta(T_n)$ for $d \geq 2$.

$1 \leq j < k-1$

Both (a) and (b) of the algorithm give a contradiction, so we assume (c) applies. Therefore, the set of "bad" vertices D has at most $k-1$ vertices, the sum of the degrees of these vertices is at least $n-|S|$, and S' has at least $M(j)$ vertices.

Consider the $k-j$ of these vertices which have the largest degrees. The claim is that each of these vertices must have degree at least $|S|$. If not, then the sum of the degrees of the $k-j-1$ largest degree vertices would be at least $n-(j+$

1) $|S|$, and some vertex of T_n would have degree at least $(n-(j+2)|S|)/(k-j-1)$, a contradiction for $d \geq j+2$.

Let $\{v, v_1, v_2, \ldots, v_{j-k-1}\}$ be any set of $k-j$ vertices of T_{n-q} which includes v and such that each has degree at least $|S|$. Let T' be the subtree of T_{n-q} spanned by these vertices. Since the length of suspended paths and the number of independent end-edges is bounded by some function of k and l, the order of the tree T' is also bounded by a function depending only upon k and l. For any embedding τ of T' into $\langle R\rangle$ which voids S and with $\tau(v)=w$, there is a $k-j-1$ set $Y=\{\tau(v_1), \tau(v_2), \ldots, \tau(v_{j-k-1})\}$ of vertices in $V(G)-S$.

If the union of the neighborhoods in $\langle R\rangle$ of the $k-j-1$ vertices Y contain all of the vertices of S except for possibly $M(j)$, then the embedding τ can be extended to T_{n-q} using all but possibly $M(j)$ of the vertices of S. Thus clearly, τ can be extended to T_n, a contradiction. Thus, we assume that there are at least $M(j)$ vertices of S adjacent in $\langle B\rangle$ to each vertex of Y.

Since $\delta(\langle R\rangle) \geq (1-p)n$, there are many embeddings τ of T' into $\langle R\rangle$ avoiding S and with $\tau(v)=w$. In fact, the number of different $k-j-1$ subsets Y yielded by such embeddings is $b \cdot n^{k-j-1}$ for some positive constant b. Each vertex of each of these subsets Y is adjacent in $\langle B\rangle$ to at least $M(j)$ vertices of S.

Consider the bipartite graph L with the vertices in the first part being the $(k-j-1)$-subsets of $V(G)-S$, and the vertices in the second part being the k-subsets of S. If all of the edges between the $(k-j-1)$-subset and the k-subset are in $\langle B\rangle$, then the corresponding vertices in L are adjacent. If some vertex in the second part of L has degree at least $\binom{l-1}{k-j-1}+1$, then $\langle B\rangle \supseteq K_{k,l}$. Since this cannot occur, we have the following inequality (the left hand side is a lower bound on the number of edges emanating from the first part, and the right hand is an upper bound on the number of edges emanating from the second part)

$$b \cdot n^{k-j-1}\binom{M(j)}{k} \leq \left(\binom{l-1}{k-j-1}\right)\binom{|S|}{k}.$$

Using the fact that $f_B(n) \leq c'' n^{(k-1)/k}$ for some constant c'', this implies that

$$(c-k)^k \leq \left(\binom{l-1}{k-j-1}\right)2^k.$$

If c is sufficiently large, this yields a final contradiction, which completes the proof of this case and the theorem. □

The same techniques used in the proof of Theorem 1 apply to special cases of bipartite graphs, in particular for even cycles.

Proof of Theorem 4. The initial observations, the nature of the induction, and the proof of the first two cases are identical to the proof of Theorem 1 with C_{2k} considered as a $B_{k,k}$ bipartite graph (i.e. $l=k$). Therefore we will use precisely the same notation used in Theorem 1 with $l=k$, and assume we are at the point

of beginning Case (3). Thus T_n has a talon with at least $n/(4k^5)$ edges. Recall that $f_B(n) \leq c' \cdot n^{1/k}$ for $B = C_{2k}$ [2].

Select p $(0<p<1)$ such that pn is the maximal degree of a talon in T_n. Thus certainly $pn \geq n/(4k^5)$. Let v be the center of this talon, and T_{n-m} the tree obtained from T_n by deleting the q edges of the talon, where $q = pn$. Also, let w be a vertex of maximal degree in $\langle R \rangle$, and S the vertices adjacent to w in $\langle B \rangle$. Since $\Delta(\langle R \rangle) \geq N - 2 \cdot f_B(n)$, S has at most $2 \cdot f_B(n)$ vertices.

Let T' be the tree obtained from T_{n-q} by deleting all of the vertices of degree 1. Since the length of suspended paths and the number of independent end-edges is bounded by some function of k and l, the order of the tree T' is also bounded by a function depending only upon k and l. Hence, there is an embedding τ of T' into $\langle R \rangle$ with $\tau(v) = w$ and $\tau(T')$ disjoint from S. In fact, there is such an embedding which avoids not only S but any $c''n$ vertices not in S as long as, for example, $cn \leq (1-p)n/2$.

If the embedding τ can be extended to T_{n-q} using all of the vertices of S except for possibly c, then it can clearly be extended to T_n. Thus, we assume that the embedding cannot be so extended, so there is a vertex not in S which is adjacent in $\langle B \rangle$ to at least c vertices of S. This can be repeated $c''n$ times to obtain a set A of $c''n$ vertices, each of which is adjacent in $\langle B \rangle$ to at least c vertices of S.

Consider the bipartite subgraph L of $\langle B \rangle$ induced by the parts A and S. In L, each vertex of A has degree at least c relative to S, $c \leq |S| \leq c' \cdot n^{1/k}$, and $|A| = c''n$. Therefore by a result in [11], there is a path in L of length $2k-2$ with both end-vertices in S. This path with w, which is adjacent in $\langle B \rangle$ to each vertex of S, generates a C_{2k}. This contradiction completes the proof of the theorem. □

4. Problems and comments

Two critical graphical parameters in the determination of the Ramsey number $r(S, G)$, when S is a large order sparse graph (or in particular a tree), are the order of S and the chromatic number $\chi(G)$ of G. Also, the Ramsey number $r(S, B)$ where B is a bipartite graph induced by two color classes in a $\chi(G)$ coloring of the vertices of G, appears to be an important factor in determining $r(S, G)$ [6, 8]. This is one of the motivations for working on the problems considered in this manuscript.

There are several places where the results presented could be improved; however, one is of particular interest. If T_n is a tree with only "small" degree vertices, then

$$r(T_n, B_{k,l}) = n + c$$

for a sufficiently large c. It would be nice to show that $c = k - 1$ is sufficient in general. For special classes of graphs this has been verified in [4] and [9].

There are several papers dealing with the Ramsey number of a fixed graph and

a sparse graph [3–5, 8]. It would be of interest to know which sparse graphs could replace the trees of Theorem 1 and Theorem 5 without altering the results.

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ON CONJECTURES OF GRAFFITI

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1.

Graffiti is a computer program which makes graph-theoretical conjectures. It was written in 84–85 and was preceded by a similar but less complex program, Little Paul, on which the author collaborated with Shui-Tain Chen.

The basic idea of Graffiti is that it "knows" certain graphs and it is capable of evaluating certain formulas formed from graph-theoretical invariants. If none of the graphs with which Graffiti is familiar is a counterexample to a formula then the formula is considered to be a conjecture. At present Graffiti is capable of computing about 60 invariants and it performs several functions but I shall describe here only those which are relevant to conjectures described in the next section. The conjectures are of the form $I \leq J$, $I \in J + K$ and $I + J \in K + L$, where literals run over 20 distinct selected invariants plus a constant invariant 1. They are respectively called conjectures of the first, second and the third type.

After the first run of the program the total number of conjectures was about 7500. The Library of the program had at the time about 40 graphs.

The number of conjectures, particularly those which are completely trivial, is the main problem and more than half of the program consists of various heuristics whose purpose is deletion of trivial and otherwise noninteresting but true conjectures. The conjectures in the next section were obtained by procedures IRIN and CNCL. IRIN deletes those conjectures which by transitivity follow from others.

All conclusions that Graffiti draws are based exclusively on graphs it knows, so "follows" means of course "follows as far as the graphs in the library of program are concerned". During the first run IRIN reduced the total number of conjectures to about 1700.

CNCL deletes those conjectures of the second and third type in which one of the invariants on the left is always smaller than an invariant on the right. This procedure may undoubtedly remove a number of interesting conjectures but it is the only one which seems to operate completely independently from IRIN.

During the first run CNCL following IRIN reduced the list of conjectures to about 140. For other sets of 20 invariants the number was usually at least 100

higher and that is one of the main reasons why I decided on this particular set of invariants.

At this point I divided conjectures in four categories: False, Open, Trivial and Proved. The last two types were stored in separate files and after repeated runs they would be automatically removed from the new lists of conjectures. After this the last three files could be updated.

For each false conjecture I had to inform Graffiti about a counterexample; Graffiti is capable of defining certain types of graphs and of performing some operations on graphs leading to more complex examples.

After the first run there were two conjectures which were already proved before. I disproved about 15 using five counterexamples and I classified about 40 conjectures as trivial.

During the following month I added to the Library of Graffiti about 40 graphs but they were counterexamples to about 80–100 conjectures which appeared in the file Open; it was not unusual for this file to grow in size after a few counterexamples were added to the library. The total number of conjectures was of course steadily going down, but if IRIN had rejected a conjecture because it had "followed" from a false conjecture then the rejected one could have reappeared after the false one has been disproved.

At the time of writing of this paper there are 6745 conjectures and the file Open contains 35 conjectures. The conjectures of the third type were not yet really studied by anyone and I decided not to include them here. There are 12 conjectures which were proved and the file Trivial contains about 80 conjectures.

The conjecture number 14 was almost completely proved by William Waller.

Graffiti has procedures computing chromatic number, independence and size of the matching but they are so slow for larger graphs (some of the counterexamples have up to 80 vertices) that I choose rather to inform the program about their values. That was particularly essential when I experimented with various graphs and Graffiti was verifying if they were counterexamples to a conjecture.

2.

We are going to use the following notation: If $R=(r_1, r_2, \ldots, r_n)$ is a real-valued vector then the value of the component which occurs most often is called the *mode* of R and the *maximal frequency* of R is the number of occurrences of the mode. In the case of a tie the mode is the largest component.

The Degree denotes the degree sequence of a graph. Hence, for example maximal frequency of Degree of a path with 3 vertices is 2 and the mode is 1; the maximal frequency of distance is the distance which occurs the maximum number of times. The *temperature* of a vertex is $d/n-d$ where d is the degree of the vertex and n is the number of vertices. Vectors can be considered random variables on a uniform space; this gives rise to some invariants defined by means

of the average value and the variance. However, the *average distance* is the mean of the vector is the expected distance between distinct vertices.

A vertex c of a graph is *central* if every other vertex of G can be reached from c in a minimum number of steps. The number of central vertices is the *center* of G and the number of steps is the radius. The maximum distance between any two vertices is the *diameter* and the *boundary* consists of those vertices whose distance from some vertex equals to the diameter.

The inverse degree is $\sum 1/d_i$ where d_i is the degree sequence. The *Randic Index*, or Randic, is $\sum (d_i d_j)^{-\frac{1}{2}}$ where the summation extends over pairs of adjacent vertices. The last two invariants are not defined in the presence of isolated vertices and of course metric invariants apply only to connected graphs.

Independence and *matching* denote respectively the size of the largest independent set and the size of the largest matching.

Finally the *rank* is the rank of the adjacency matrix, and the *zenith* is the number of vertices of maximum degree.

The following conjectures are now in the file Open:

1. average temperature $\leqslant$ rank
2. average temperature $\leqslant$ variance of degree + maximal frequency of degree
3. inverse degree $\leqslant$ Randic index + maximal frequency of degree
4. mode of distance $\leqslant$ radius + Randic index
5. mode of distance $\leqslant$ average distance + Randic index
6. mode of distance $\leqslant$ matching + average distance
7. radius $\leqslant$ zenith + maximal frequency of degree
8. radius $\leqslant$ variance of degree + maximal frequency of degree
9. radius $\leqslant$ Randic index + average temperature
10. radius $\leqslant$ Randic index + variance of degree
11. radius $\leqslant$ average distance + Randic index
12. radius $\leqslant 1$ + Randic index
13. average distance $\leqslant$ Randic index
14. average distance $\leqslant$ independence
15. average distance $\leqslant$ variance of degree + maximal frequency of degree
16. average distance $\leqslant$ variance of degree + inverse degree
17. average distance $\leqslant$ mode of distance + inverse degree
18. chromatic number $\leqslant 1$ + rank

Below are listed conjecture which were proved so far. The 6th conjecture was proved by Shui-Tai Chen and the 7th by William Waller. The 10th was proved by Chen and myself. Sometimes they resulted in stronger theorems, but they are stated here in the form in which they were conjectured.

1. radius $\leqslant$ independence
2. average temperature $\leqslant$ chromatic number

3. diameter $\leqslant$ rank
4. radius $\leqslant$ matching
5. average distance $\leqslant 1 +$ matching
6. inverse degree $\leqslant$ independence + Randic index
7. inverse degree $\leqslant$ matching + independence
8. Randic index $\leqslant$ matching + independence
9. Randic index $\leqslant$ number of vertices
10. average degree $\leqslant$ matching + Randic index
11. Randic index $\leqslant$ matching + inverse degree
12. matching $\leqslant$ Randic index + inverse degree

None of these theorems are very difficult, but almost every one, I think, is of some interest.

For example, the second theorem on the list is one of the easiest but it provides, what I consider a neat lower bound for the chromatic number χ. It can be proved as follows: Let C be a good χ-coloration of G and let $\bar{G}$ be a complete χ-partite graph compatible with this coloration. Because the average temperature of $\bar{G}$ is greater than that of G it is enough to prove the theorem for $\bar{G}$. But the average temperature of $\bar{G}$ is $\chi - 1$.

Hence we proved that the average temperature is $\leqslant \chi - 1$ and the equality holds true iff G is complete k-partite graph.

This conjecture realized my hope that Graffiti will produce interesting conjectures which are easy to prove. But beforehand I did not see too many justifications for this hope. The only reason I could think of was that Graffiti might find overlooked facts and those should be independent from both their difficulty and the significance.

On the other hand I expected Graffiti to find conjectures difficult to prove. I thought that the program would operate like a statistician who finds a correlation without a cause. Perhaps some conjectures which Graffiti has already made are difficult to prove but in most of the cases one can clearly see a reason for making the conjecture.

A few of the counterexamples were found by Chen and Waller and a few are based on an idea of Erdös and Spencer dealing with the disproved conjecture average distance $\leqslant$ inverse degree.

The graphs represented in the Fig. 1 are fairly typical in that each has an extreme value for one of the invariants. This is the case with the mode of distance of Crab, the average distance of Barbells and Binaries and the Randic Index of Milky Way. Finally, in Milkweeds the radius is almost as large as the diameter.

Some of these graphs are fairly large but the reader may notice that for all it is relatively easy to write the defining programs. Thus so far I have not had to resort to elaborate lising of edges to define a graph. But a close call came when Chen found an 8-vertex counterexample to the conjecture radius $\leqslant$ maximal frequency

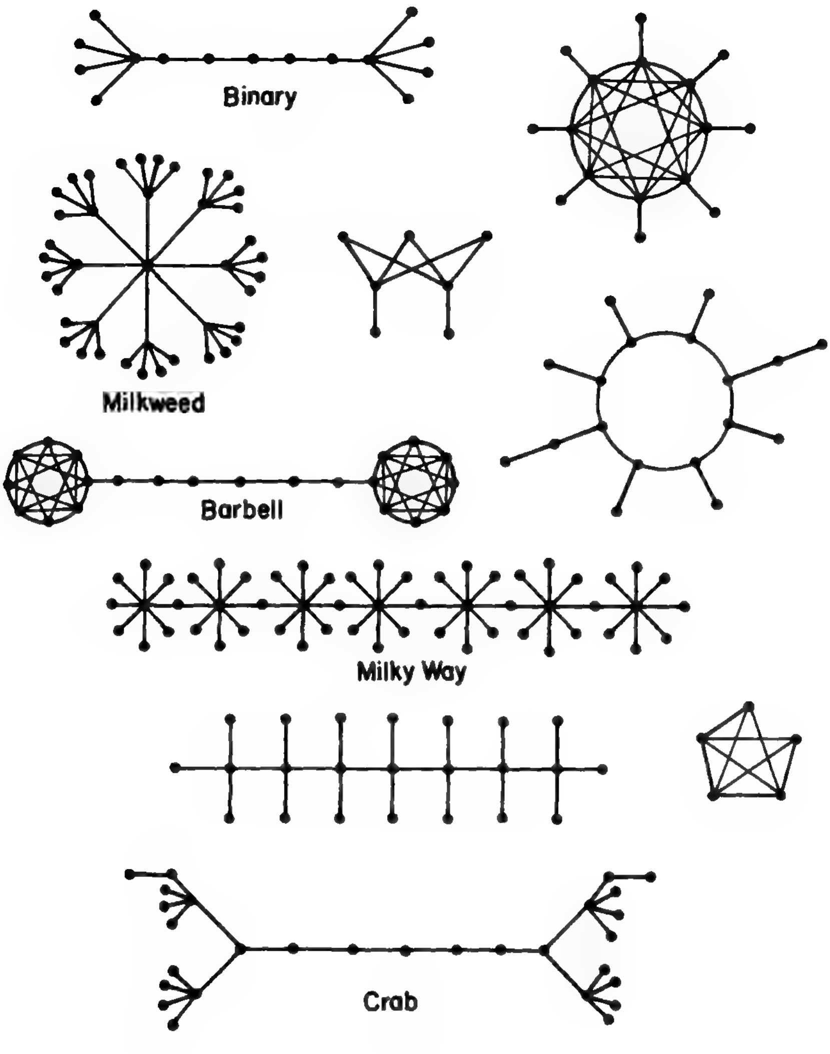

Fig. 1.

of degree. Eventually I wrote a procedure which automatically searches for counterexamples to conjectures of this type by optimizing the polarity. The procedure has since found several other useful graphs.

In the future I will probably use more such procedures. There are many ways of doing this and I may even already have a prototype for one. In 1982 Noriko Naumann wrote as a part of her master's thesis a short but scintillating program called Pythia which guesses the next number which should appear in a sequence of integers. The program guesses patterns by simplfying them in a manner similar to the one described by Dewdney in the April 1986 issue of Scientific American.

It can be adapted to graph theory and it has both conjecture-making and example-finding potential.

I would like to thank Robert Cottingham for writing for me a fast program deleting the text of one file from another.

Note added in Proof. Conjectures 14 and 1 were proved respectively by Fan Chung and myself. Conjecture 7 was refuted by Shui-Tan Chen.

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SMALL ORDER GRAPH-TREE RAMSEY NUMBERS

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With but a few exceptions, the Ramsey number $r(G, T)$ is determined for all connected graphs G with at most five vertices and all trees T.

1. Introduction

For graphs G and H define the *Ramsey number* $r(G, H)$ as the least number N such that in each two-coloring $(R, B) = $ (red, blue) of the edges of K_N there is a red copy of G or a blue copy of H. Sometimes we start with a graph L on N vertices and refer to L as the red graph and to its complement $\bar{L}$ as the blue graph.

We investigate $r(G, H)$ when G is any connected graph with at most five vertices and H is an arbitrary tree. All but a few of the numbers are obtained. In the process of the investigation close attention is given to the graphical parameters of G and H which affect the Ramsey number.

Similar Ramsey numbers have been considered in papers [1–4, 7–9], when G is a fixed graph and H is a large sparse graph (one with many vertices and few edges). Of course a large order tree is one such sparse graph. Complications arise in finding this Ramsey number when H is of arbitrary order or has large maximal degree, even when H is a tree and G is of small order.

To avoid lengthy case analysis type arguments most of the Ramsey numbers discussed are given without proof. Instead emphasis is given to the general strategy of the proofs and how the results relate to known Ramsey numbers.

2. Terminology, notation, and related results

For the most part the terminology and notation used conforms with the usual and accepted. Specialized terminology is summarized in what follows.

For a graph G, $p(G)$ denotes the number of vertices of G and $s(G)$ (called the *chromatic surplus*) is the minimum number of vertices in a color class under all $\chi(G)$ – vertex colorings of G. If H is a subgraph of G, then $G - H$ denotes the

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graph obtained by deleting the edges of H. For disjoint graphs G and H, the *join* $G+H$ is the graph obtained by adding the missing edges between vertices of G and vertices of H. The symbol tG denotes t disjoint copies of the graph G.

In order to conveniently describe trees which contain large stars as subgraphs we develop a special notation. Let F be a forest with each tree in the forest rooted, say with roots $v_1, v_2, \ldots, v_r$, and $K(1, s_1)$ a star with vertices $u_0, u_1, \ldots, u_s$ and u_0 the center of the star. If $s \geq r$, a new tree is found by identifying u_i and v_i for $1 \leq i \leq r$. This tree is denoted by $T_n(F)$ where n is the total number of vertices in the tree. In the forest F unmarked stars (paths) are assumed rooted at their centers while ones marked with an asterisk are rooted at a vertex of degree one. For example, for the forests $F_1 = K(1,2) \cup P_3^*$ and $F_2 = (K(1,3))^* \cup P_4$, $T_{10}(F_1)$ and $T_{10}(F_2)$ are shown in Fig. 1. Throughout the paper the symbol T_n is used exclusively to denote an arbitrary tree on n vertices.

Let G and H be graphs with H connected such that $p(H) \geq s(G)$. Then $r(G,H) \geq (\chi(G)-1)(p(H)-1)+s(G)$ in view of the example in which the red graph is the complete multipartite graph with $\chi(G)-1$ parts of size $p(H)-1$ and one of size $s(G)-1$. If $r(G,H) = (\chi(G)-1)(p(H)-1)+s(G)$ then H is said to be *G-good*. One of the earliest results in generalized Ramsey theory was that of Chvátal [6], in which he stated that $r(K_m, T_n) = (m-1)(n-1)+1$ for all $m, n \geq 1$, i.e. all trees are K_m-good. Thus for graphs of this paper, G of small order and H an arbitrary tree, the chromatic number of G and the order of H are parameters which affect $r(G,H)$.

It is known that

(1) a connected sparse graph H of sufficiently large order and appropriate maximal degree is G-good for G a fixed graph [7],

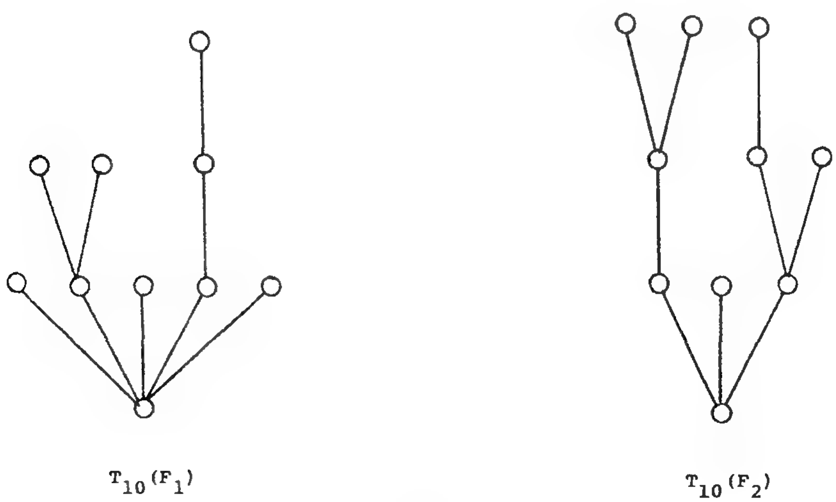

Fig. 1.

(2) a tree T_n is $K(1, 1, m_1, m_2, \ldots, m_k)$-good for $m_1, m_2, \ldots, m_k$ fixed and n large [3], and

(3) for large n the star $K(1, n)$ is neither $K(1, m_1, m_2, \ldots, m_k)$-good for $m_k \geqslant m_{k-1} \geqslant \cdots \geqslant m_1 \geqslant 2$ (each m_i fixed), nor $K(2, 2)$ good [4, 9].

In fact it is shown in [9] that $r(K(2, 2), K(1, n)) > n + \lfloor n^{\frac{1}{2}} - 6n^{\frac{3}{10}} \rfloor$ for n large, and that $r(K(2, 2), T_n) = \max\{4, n+1, r(K(2, 2), K(1, \Delta(T_n)))\}$ for all n. We shall see that the only troublesome small graphs G—those where $r(G, T_n)$ is not always determined—are those which contain $K(2, 2)$ as an induced subgraph.

We record (for later reference) in Theorem A two goodness results. Clearly both are generalizations of Chvatál's theorem.

Theorem A. (1) *If $l, m \geqslant 1$, $n \geqslant 2$ and $l > (m-1) - \lfloor (m-1)/(n-1) \rfloor (n-1)$, then*

$$r(K_l + \bar{K}_m, T_n) = \left(l + \left\lfloor \frac{m-1}{n-1} \right\rfloor\right)(n-1) + 1. \quad [10]$$

(2) *If $m \geqslant 6$, $n \geqslant 3$ and $T_n \neq K_{1,n-1} (n \geqslant 4)$, then $r(K_m - tK_2, T_n) = (m-t-1)(n-1) + 1$ for all t, $0 \leqslant t \leqslant \lfloor (m-2)/2 \rfloor$.* [11]

We wish to see which other graphical parameters of G and T_n affect $r(G, T_n)$. First let t_1 and t_2 be positive integers and set $L = t_1 K_{p(G)-1} \cup t_2 K_{p(G)-2}$. Choose t_1 and t_2 such that T_n is not a subgraph of $\bar{L}$ and such that $p(L)$ is maximal. Setting $t(G, T_n) = p(L)$ it is clear that $r(G, T_n) \geqslant t(G, T_n) + 1$.

As innocent as this bound appears it frequently is the value of $r(G, T_n)$. We cite two such cases which are recorded formally in Theorem B below. To do this we need additional notation. Let $\alpha(G)$ denote the independence number of G. For the tree T_n, let $\alpha'(T_n) = \min\{\alpha(F) \mid F$ is the forest obtained by deleting from T_n a vertex and its neighbors$\}$. Thus $\alpha'(T_n)$ is a measure of how small the independence number of a non-neighborhood of a vertex of the tree T_n can be. One can for example show that if $2\alpha' + 3 \leqslant k$, then $n + k - 2 - \alpha'(T_n) - \delta = t(P_k, T_n) + 1$ where $\delta = 0$ if $k-1$ divides $n + k - 3 - \alpha'(T_n)$ and $\delta = 1$ otherwise. In fact a graph L which works in this case can be obtained as follows: let $n + k - 3 - \alpha'(T_n) - \delta = a(k-1) + b$ where $0 < b \leqslant k-1$ and set $t_1(P_k, T_n) = a - k + 2 + b$ and $t_2(P_k, T_n) = k - 1 - b$. It can be checked that the resulting L satisfies the requisite condition, (see [1]).

Theorem B. (1) *Let H be a connected graph with n vertices and no more than $n(1 + 1/81k^5)$ edges. Then for $k \geqslant 2$ and $n \geqslant 352k^{12}$*

$$r(P_k, H) = \max\{n + \lfloor \tfrac{1}{2}k \rfloor - 1,\ n + k - 2 - \alpha'(H) - \delta\}$$

where $\delta = 0$ if $k-1$ divides $n - k - 3 - \alpha'(H)$ and $\delta = 1$ otherwise. [1]

(2) *Let k be an integer $\geqslant 2$, and $n \geqslant 2(3k-2)(2k-3)(k-2)+1$. Then $\max\{n, n + k - 1 - \alpha'(T_n) - \delta\} \leqslant r(K(1, k), T_n) \leqslant \max\{n, n + k - 1 - \alpha'(T_n)\}$ where $\delta = 0$ if $n - k - 2 - \alpha'(T_n)$ is divisible by k and $\delta = 1$ otherwise.* [8]

The remaining relevant graphical parameter of G and $T_n(p(G) \leq 5)$ which affects the value $r(G, T_n)$ is the maximal degree Δ of both G and T_n. It is well known (see [5]) that $r(K(1, s_1), K(1, s_2)) = s_1 + s_2 - \epsilon(s_1, s_2)$, where

$$\varepsilon(s_1, s_2) = \begin{cases} 1 & \text{when both } s_1 \text{ and } s_2 \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

One expects $r(G, T_n) = r(\Delta(G), \Delta(T_n))$ for certain special graphs G and special trees T_n.

Having explored the important grapical parameters affecting the Ramsey number we are prepared to define the function f which (we shall see) does for the most part satisfy $r(G, T_n) = f(G, T_n)$ when G is of small order. Let G and H be graphs, H connected, and define

$$\begin{aligned} f(G, H) = \max\{&(\chi(G)-1)(p(H)-1) \\ &+ s(G), t(G, H)+1, \Delta(G)+\Delta(H)-\varepsilon(\Delta(G), \Delta(H))\} \end{aligned}$$

From the above discussion it is clear that $f(G, T_n)$ is a lower bound for $r(G, T_n)$ and that there are pairs of graphs where each of the values maximized in f dominants the remaining ones.

3. Main results

With the important groundwork covered we are in the position to state the results. Before doing so a comment or two should be made about the value of $t(G, H)$. For arbitrary graphs this value is difficult, if not impossible, to compute. In the case where it is needed in the theorems below such is not the case. Surely when $\chi(G) \geq 3$ and H is a tree T_n, except possibly for $n = 1$ or 2, $f(G, T_n) > t(G, T_n) + 1$. Thus the reader will only need to be concerned with $t(G, T_n)$ when G is bipartite. More importantly it can be shown that in all cases considered in Theorem 1 (below) $t(G, T_n)$ only affects the value of $f(G, T_n)$ when $\Delta(T_n) \geq n - 5$. Using this makes $f(G, T_n)$ easy to calculate.

Theorem 1. *Let G be a connected graph with at most five vertices and let T_n be a tree on n vertices, $n \geq 9$. Then with but a few exceptions (those listed in (i)–(iv) below) we have $r(G, T) = f(G, T_n)$.*

(i) $r(K_5 - 2K_2, K(1, n-1)) = 2n + 1$ *when n is even.*
(ii) $r(K_5 - P_4, K(1, n-1)) = 2n$ *when n is even*
(iii) $r(K(2, 2), T_n) = r(K(2, 3) - e, T_n) = \max\{n + 1, r(K(2, 2), \Delta(T_n))\}$
(iv) *There exists a constant c such that* $n + 1 \leq r(K(2, 3), T_n) \leq \max\{n + c, r(K(2, 3), \Delta(T_n))\}$.

The lack of exactness in items (iii) and (iv) of the theorem stems primarily from

not knowing $r(G, K(1, n))$ exactly when G is bipartite. It is known that $n + \lfloor n^{\frac{1}{2}} - 6n^{\frac{3}{10}} \rfloor < r(K(2, 2), K(1, n)) \leqslant n + \lceil n \rceil + 1$ for n large [7, 9].

Theorem 2. *If T_n is a tree with $n \leqslant 8$ and G a connected graph with $p(G) \leqslant 5$, then $r(G, T_n) = \max\{f(G, T_n), f(T_n, G)\}$, except for the pairs (G, T_n) given in Tables* 1 *and* 2.

Since there are thirty-one connected graphs with at most five vertices and forty-eight trees with at most eight vertices the number of exceptions given in the tables is small. We will not prove Theorem 2 here, since the argument is tedious and lengthy. Also only part of the proof of Theorem 1 will be given for the same reason.

Partial Proof of Theorem 1. We only concern ourselves with the proof when $\chi(G) \geqslant 3$. The reason is twofold. First the proof for those cases when G is a tree and n is small has many special subcases and is similar in nature to the proof of Theorem B. Thus little is gained by presenting this part of the proof. The only remaining possibility is when $\chi(G) = 2$ and $K(2, 2)$ is a subgraph of G. The proof of (iii) and (iv) is essentially in [9], so neither is repeated here.

Therefore assume $\chi(G) > 2$ so that $f(G, T_n) = (\chi(G) - 1)\ (n - 1) + 1$. When $G = K_5$, K_4, or $K_5 - e$ the value of $r(G, T_n)$ is given in Theorem $A(1)$. Also clearly $r(K_4, T_n) \leqslant r(K_5 - P_3, T_n)$, $r(K_5 - K(1, 3), T_n) \leqslant r(K_5 - e, T_n)$ and by Theorem $A(1)$ $r(K_4, T_n) = r(K_5 - e, T_n) = 3n - 2$, $n \geqslant 3$. Thus the theorem holds when $\chi(G) \geqslant 4$.

The only remaining possibility, when $\chi(G) = 3$, is handled as follows. First $r(K_3, T_n) = r(K_4 - e, T_n)$ by Theorem A(1) and $r(K_4 - e, T_n) \geqslant r(K_3 + e, T_n) \geqslant r(K_3, T_n)$. Each remaining 3-chromatic graph G is a subgraph of one of $K_5 - 2K_2$, $K_5 - P_4$, and $K_5 - K_3$. But $r(G, T_n) \geqslant 2n - 1$ for each such G and we show in the

Table 1.

	$K(2, 2)$	$K(1, 4)$	$K(2, 3) - e$	K_5–K_3	K_5–P_4	K_5–$2K_2$
$K(1, 3)$	6			8	8	9
$K(1, 5)$	8				12	13
$T_7(3K_2)$		8				
$K(1, 6)$	9					
$T_7(K(1, 2))$		8				
$K(1, 7)$	11		11		16	17
$T_8(K(1, 3))$		9				

Table 2.

	$K(2, 3)$
$K(1, 3)$	7
$T_5(K_2)$	7
$K(1, 4)$	8
$T_6(P_3^*)$	8
$T_6(K_2)$	8
$T_6(K(1, 2))$	8
$K(1, 5)$	10
$T_7(K_2)$	10
$K(1, 6)$	11
$T_8(P_3^*)$	10
$T_8(2k_2)$	10
$T_8(K_2)$	11
$K(1, 7)$	13
$T_8(K(1, 3)^*)$	10
$T_8(K(1, 2))$	10
$T_8(K(1, 3))$	10

lemma given below (for $n \geq 6$) that $r(K_5 - 2K_2, T_n) = r(K_5 - P_4, T_n) = r(K_5 - K_3, T_n) = 2n - 1$ when $T_n \neq K(1, n-1)$ for n even. Thus upon proof of the lemma the 3-chromatic case is proved except when $T_n = K(1, n-1)$ with n even. It is easy to check for $\chi(G) = 3$ and n even that $r(G, K(1, n-1)) = 2n - 1$ unless $G = K_5 - P_4$ or $K_5 - 2K_2$ and these are considered in the lemma.

Lemma. *For $n \geq 6$ each of the following hold.*
(i) $r(K_5 - K_3, T_n) = 2n - 1$.

(ii) $r(K_5 - 2K_2, T_n) = \begin{cases} 2n+1 & \textit{when } T_n = K(1, n-1),\ n \textit{ even} \\ 2n-1 & \textit{otherwise.} \end{cases}$

(iii) $r(K_5 - P_4, K(1, n-1)) = 2n$ *for n even.*

Proof (i) Surely $r(K_5 - K_3, T_n) \geq 2n - 1$ so let each edge of K_{2n-1} be colored red or blue. We suppose there is such a coloring where there is no red $K_5 - K_3$ and no blue T_n and show this leads to a contradiction.

First let $T_n = K(1, n-1)$. Since the two colored K_{2n-1} contains no blue

$K(1, n-1)$, each vertex has red degree at least n. But $r(K_3, T_n) = 2n - 1$ so that the red graph contains a K_3 whose vertices we label v_1, v_2, v_3. If for $i \neq j$, v_i and v_j have two common red adjacencies off the red triangle, then the red graph contains a $K_5 - K_3$. Since this cannot happen, there are at least $3 + (n-2) + (n-3) + (n-4) = 3n - 6$ vertices in K_{2n-1}. Hence $n \leq 5$, a contradiction.

Next assume $T_n \neq K(1, n-1)$. Since Theorem 1 is assumed true when $\chi(G) = 2$, $r(K(1, 3)T_n) \leq n + 1$. Thus if the red graph in the two colored K_{2n-1} has a vertex of degree $n + 1$, its neighborhood contains a red $K(1, 3)$ implying the existence of a red $K_5 - K_3$. Hence each vertex of the K_{2n-1} has blue degree at least $n - 2$, and since $T_n \neq K(1, n-1)$, the blue graph either contains a T_n or a K_{n-1}. Since the former is impossible, the existence of a blue K_{n-1} and no blue T_n implies the red graph contains a $K(n-1, n)$. This forces either a red $K_5 - K_3$ or a blue T_n, a final contradiction.

(ii) We first consider the case when $T_n = K(1, n-1)$ with n even. To see $r(K_5 - 2K_2,\ K(1, n-1)) > 2n$ let L (the red graph) be the graph formed by placing a perfect matching in each part of a complete bipartite graph $K(n, n)$. Clearly this graph contains no $K_5 - 2K_2$ and the blue graph $\bar{L}$ contains no $K(1, n-1)$. Thus color each edge of K_{2n+1} red or blue and suppose the coloring is such that there is no red $K_5 - 2K_2$ and no blue $K(1, n-1)$. We consider two possibilities, the first of which is that the red graph contains a K_4. Let S denote the set of those vertices of K_{2n+1} not part of the red K_4. No red $K_5 - 2K_2$ implies each vertex of S is adjacent in red to at most two vertices of the K_4. Thus the total number of blue edges incident to the four vertices of the K_4 is at least $2(2n-3)$ implying that one of these four vertices must have blue degree at least $n - 1$, a contradiction.

Since by Theorem A $r(K_4 - K_2, T_n) = 2n - 1$, the two colored K_{2n+1} contains a red $K_4 - K_2$ and as just argued no red K_4. But by repeating the above argument where the red $K_4 - K_2$ replaces the red K_4, and S is the set of vertices of K_{2n+1} not part of the red $K_4 - K_2$, we again obtain a blue $K(1, n-1)$. This contradiction completes this part of the roof.

Next consider the remaining possibility, when T_n is a tree that is not a star with an odd number of edges. We then show $r(K_5 - 2K_2, T_n) = 2n - 1$ for $n \geq 3$. Since $2n - 1$ is clearly a lower bound, we need only show $r(K_5 - 2K_2, T_n) \leq 2n - 1$, which is done by induction. It is easy to check that this holds for $n = 3$ and for $n = 4$ when $T_4 \neq K(1, 3)$. Thus assume the result for values less than n and consider a K_{2n-1} with each edge colored red or blue. We consider two cases.

Case 1. The red graph contains a K_4

Delete from T_n two end vertices x and y such that the resulting tree $T_{n-2} = T_n - x - y$ is not a star when $T_n \neq K(1, n-1)$ and $n \geq 6$. By induction the two

colored K_{2n-5} obtained from the two colored K_{2n-1} by deleting the red K_4 contains the blue T_{n-2}. let u and v denote the neighbors of x and y in T_n. Now u and v are blue adjacent to distinct vertices of the red K_4 (when $u=v$ this means this vertex has two blue adjacencies to the red K_4) or one of them has at least three red adjacencies to the red K_4. Hence there either is a red K_5-2K_2 or a blue T_n in the two colored K_{2n-1}.

Case 2. Since $r(K_4-K_2, T_n)=2n-1$, the red graph contains a red K_4-K_2 and no red K_4

The argument for this parallels the one given in Case 1, simply repeat the argument given with the red K_4-K_2 replacing the red K_4.

(iii) To see $r(K_5-P_4,\ K(1,n-1))>2n-1$ let L (the red graph) be the one obtained from the complete bipartite graph $K(n,n-1)$ by inserting a perfect matching in the part with n vertices. Clearly this graph contains no K_5-P_4 and the blue graph $\bar{L}$ contains no $K(1,n-1)$.

Two color the edges of a K_{2n} and suppose this graph contains no red K_5-P_4 and no blue $K(1,n-1)$. For $n=4$ it is easy to verify this is impossible. Hence assume n is even, $n>4$, and proceed by induction or n. Just as noted earlier the red graph contains a K_4-K_2. Let S be the remaining $2n-4$ vertices of the K_{2n} (other than those four of the K_4-K_2). By the induction assumption some vertex of S has blue degree at least $n-3$ relative to the graph induced by the set S. Also each vertex of S must be blue adjacent to at least two vertices of the red K_4-K_2 to avoid a red K_5-P_4. This makes some vertex of S have blue degree at least $n-1$, a contradiction. This completes the proof of the lemma. $\square$

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ENUMERATING PHYLOGENETIC TREES WITH MULTIPLE LABELS

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Evolutionary trees are used in biology to illustrate postulated ancestral relationships between species and are often called phylogenetic trees. They can be characterized in graph theoretic terms by certain classes of labelled trees. Disjoint subsets of the labelling set are assigned to tree vertices so that all pendant vertices and any vertices of degree two are labelled. Here we determine exact and asymptotic numbers for two classes of trees in which multiple vertex labels are allowed. In the first class vertices of degree two are forbidden and in the second class vertices of degree greater than two cannot be labelled. A general method is presented for deriving the asymptotic analysis of any multiple label case. Asymptotic results for the two classes of trees under study are then obtained by applying this method to previously published results. This paper completes work by the authors on the enumeration of various classes of phylogenetic trees.

1. Introduction

Biologists often represent postulated evolutionary relationships between existing biological species by means of a tree. Such a diagram, linking related species to a common ancestor, is called a *phylogenetic tree* or *phylogeny*.

In this paper we define a *phylogeny* to be a tree in the graph theoretic sense together with a set $\{1, 2, \ldots, n\}$, of *labels* and a function f, mapping the labels into the vertex set of the tree. Every vertex of degree less than three in the tree must be in the image of f. Note that some vertices may possess multiple labels and some vertices no label at all. The label set corresponds to a given set of existing species. The *magnitude* of a phylogeny is the number n of its labels. The *order* of a phylogeny is the number of vertices in it. A *planted phylogeny* is a phylogeny having a pendant vertex which is distinguished, and is termed its *root*. This root represents the common ancestor of all the species in the labelling set and thus is not given one of the labels of this set. The construction and the biological significance of phylogenies have been discussed by many authors. See Penny et al. [11] for a recent exposition. The graph theoretic notation and terminology used in this paper is fully defined in the book by Harary [7].

The above mentioned paper by Penny et al. discusses the determination of

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phylogenies which satisfy the optimality criteria of various scientific models of evolution. In the search for phylogenies which are optimal in this sense it is of interest to know how many possible feasible phylogenies exist for a given set of n labelled species. The exact and asymptotic numbers of phylogenies with given magnitude was calculated by the present authors in [3]. The mean and variance of the orders of these trees was also determined as well as the effect of requiring that each label be a singleton. The authors carried out the same analysis for binary phylogenies in [4]. The effect of restricting the labelling to pendant vertices was also studied. In [5] this analysis was carried out for phylogenies without vertices of degree two and where only pendant vertices are labelled. Once again the effect of making all the labels singletons was studied. In [6] the analysis was repeated for phylogenies which are labelled only with singletons and where there are no vertices of degree two. A second case was also analysed in which there are no restrictions on vertex degree, all nonempty labels are still singletons, but only vertices of degree one or two can be labelled. The four papers just referenced comprise an analysis of 10 classes of phylogenies. There are two remaining cases which complete the set of all feasible combinations of restrictions on vertex degree, labelling, and label size:

Case 1. There are no vertices of degree two, and interior vertices may possibly be labelled.

Case 2. There are no restrictions on vertex degree, and only vertices of degree one or two can be labelled.

In each of these cases, labels need not be singletons. In the biological context, this corresponds to the case where different species cannot be distinguished by the comparison criterion being used to construct the phylogeny.

The objectives and terminology which are common to the two cases are now introduced. We denote by T_n the number of phylogenies of magnitude n. A useful device is the exponential generating function:

$$T(x) = \sum_{n=1}^{\infty} T_n x^n / n! \tag{1.1}$$

It is also useful to calculate the mean and variance of the numbers of vertices in phylogenies of given magnitude. This is done by determining $T_{n,p}$, the number of phylogenies with magnitude n and order p. The corresponding exponential generating function is

$$T(x, y) = \sum_{n=1}^{\infty} \sum_{p=1}^{2n-2} T_{n,p} x^p y^p / n!. \tag{1.2}$$

Note that

$$T(x) = T(x, 1).$$

A proof that $1 \leq p \leq 2n-2$ is given in [2].

The numbers just discussed are first determined for planted phylogenies. We denote by P_n, the number of planted phylogenies of magnitude n and $P_{n,p}$, the number of these of order p. Further, we define the exponential generating functions:

$$P(x) = \sum_{n=1}^{\infty} P_n x^n/n! \tag{1.3}$$

$$P(x, y) = \sum_{n=1}^{\infty} \sum_{p=1}^{2n-2} P_{n,p} x^n y^n/n!. \tag{1.4}$$

We also wish to determine the μ_n, and the variance σ_n^2, of the number of vertices among all the phylogenies with magnitude n. We can achieve this by using the following recurrence relations for the first and second moments about the origin:

$$T_n^{(1)} = \sum_{p=1}^{2n-2} pT_{n,p}, \tag{1.5}$$

$$T_n^{(2)} = \sum_{p=1}^{2n-2} p^2 T_{n,p}. \tag{1.6}$$

Of course

$$\mu_n = T_n^{(1)}/T_n \tag{1.7}$$

and

$$\sigma_n^2 = (T_n^{(2)}/T_n) - \mu_n^2. \tag{1.8}$$

We define exponential generating functions for these moments:

$$T^{(1)}(x) = \sum_{n=1}^{\infty} T_n^{(1)} x^n/n!, \tag{1.9}$$

$$T^{(2)}(x) = \sum_{n=1}^{\infty} T_n^{(2)} x^n/n!$$

We note that

$$T^{(1)}(x) = T_y(x, 1) \tag{1.10}$$

and

$$T^{(2)}(x) = T_{yy}(x, 1) + T_y(x, 1). \tag{1.11}$$

Hence we can evaluate the moment generating functions by using (1.10) and (1.11).

Let

$$
\begin{aligned}
&S_n = T_n^{(2)} - T_n^{(1)},\\
&R_n = T_n^{(1)},\\
&R(x) = T^{(1)}(x), \quad \text{and}\\
&S(x) = T^{(2)}(x) - R(x).
\end{aligned}
\tag{1.12}
$$

We can compute values for $T_n^{(1)}$, $T_n^{(2)}$, S_n, μ_n, and σ_n^2 by using values for P_n, T_n, and (1.3)–(1.12).

2. Transformation from single to multiple labels

We note that Case 1 differs from another case analyzed previously by us in [6, §2] only in that multiple labels are allowed at each vertex, instead of being restricted to being at most one. Case 2 differs from that of [6, §3] in the same way. In this section we show how to derive the exact and asymptotic values of the numbers of interest in both of these cases from the available results for the singleton label cases. In fact, the assumptions required in our analysis are true also for four other cases previously studied by the authors ([3 §4, 4 §2, 4 §4, 5 §2]). Thus the results to be derived could be used to give an alternative treatment of those cases.

Let $\bar{P}$, $\bar{T}$, $\bar{R}$, and $\bar{S}$ denote the functions for the singleton label case corresponding to P, T, R, and S respectively. Whereas the exponential generating function for labelling a vertex with a single label is x, it is $(e^x - 1)$ for multiple label cases. This is because any vertex may receive any positive number of labels. For any $k \geqslant 1$ there is just one way to assign k labels to a vertex. Interleaving of label sets is accounted for in multiplying exponential generating functions together; see [8, chapter 1] for an account of the uses of exponential generating functions in labelled enumeration. Thus $P(x) = \bar{P}(e^x - 1)$, $T(x) = \bar{T}(e^x - 1)$, $R(x) = \bar{R}(e^x - 1)$, and $S(x) = \bar{S}(e^x - 1)$. From the fact that $(e^x - 1)^k/k!$ is the exponential generating function

$$\sum_{n=1}^{\infty} S(n, k)x^n/n! \tag{2.1}$$

for Stirling numbers of the second kind, it follows that

$$P_n = \sum_{k=1}^{n} S(n, k)\bar{P}_k, \quad n \geqslant 1. \tag{2.2}$$

The analogous equations hold for T_n, R_n, S_n where in these cases $n \geqslant 2$. Stirling numbers are readily calculated from the recurrence

$$S(n+1, k) = S(n, k-1) + kS(n, k) \tag{2.3}$$

which holds for $k \geq 1$, and from the boundary conditions:

$$s(0,0)=1, \qquad S(0,k)=0 \quad \text{if } k>0, \qquad S(n,0)=0 \quad \text{if } n>0.$$

We now assume that $\bar{P}(x)$, $\bar{T}(x)$, $\bar{R}(x)$, and $\bar{S}(x)$ all have the same radius of convergence $\bar{\rho}$, $0<\bar{\rho}<1$, and that for each of these series the sole singularity on the circle of convergence $|x|=\bar{\rho}$ is the positive real point $x=\bar{\rho}$ when x is viewed as a complex variable. We also assume that in some neighborhood of $x=\bar{\rho}$, an expansion of the form

$$\bar{P}(x)=a_0-a_1(\bar{\rho}-x)^{\frac{1}{2}}+a_2(\bar{\rho}-x)+a_3(\bar{\rho}-x)^{\frac{3}{2}}+\cdots \tag{2.4}$$

is valid, where $a_1>0$.

It should be clear that $\rho=\ln(1+\bar{\rho})$ is the radius of convergence of the series $P(x)$. For if $|x| \leq \rho$ then

$$\begin{aligned}|e^x-1| &= \left|x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right| \\ &\leq \rho+\frac{\rho^2}{2}+\frac{\rho^3}{6}+\cdots=e^\rho-1=\bar{\rho},\end{aligned}$$

with equality only if $x=\rho$. Thus $P(x)$ is analytic inside the circle $|x|=\rho$, and at all points on the circle except possibly for $x=\rho$. By expanding $P(x)$ in a neighborhood of $x=\rho$ it will be shown that ρ is, in fact, a singularity of $P(x)$. This establishes ρ as the radius of convergence of $P(x)$, and also as the sole singularity of $P(x)$ on its circle of convergence.

Asymptotic results can now be deduced by applying Stirling's formula, since the coefficient of x^n in $(1-x)^{-s}$ is just $\Gamma(s+n)/\Gamma(s)\ \Gamma(n+1)$ provided that $s \neq 0, 1, 2, \ldots$. By Stirling's formula this ratio can be expressed as

$$\frac{n^{s-1}}{\Gamma(s)}\left(1+\frac{s(s-1)}{2n}+O(\sqrt{n})\right) \tag{2.5}$$

for $n \to \infty$. In (2.4), then, the term $-a_1(\bar{\rho}-x)^{\frac{1}{2}}$ contributes

$$\tfrac{1}{2}a_1\Pi^{-\frac{1}{2}}\bar{\rho}^{\frac{1}{2}}n^{-\frac{3}{2}}\bar{\rho}^{(-n)}\left(1+\frac{3}{8n}+O\left(\frac{1}{n^2}\right)\right)$$

to $\bar{P}_n$. The next term, $a_3(\bar{\rho}-x)^{\frac{3}{2}}$, contributes

$$\tfrac{3}{4}a_3\Pi^{-\frac{1}{2}}\bar{\rho}^{\frac{3}{2}}n^{-\frac{5}{2}}\bar{\rho}^{(-n)}\left(1+O\left(\frac{1}{n}\right)\right)$$

when taken to the same order. The remaining terms collectively contribute $O(n^{-\frac{7}{2}}\bar{\rho}^{(-n)})$, as can be deduced from Darboux's Theorem (see Theorem 4 of Bender [1]) or from Pólya's Lemma (see [9]). Adding these up, we have

$$\frac{\bar{P}_n}{n!}\bar{\rho}^n n^{\frac{3}{2}}=\bar{A}_\rho\left(1+\frac{\bar{B}_\rho}{n}+O\left(\frac{1}{n^2}\right)\right)$$

where

$$\bar{A}_P = \frac{a_1}{2}\left(\frac{\bar{\rho}}{\pi}\right)^{\frac{1}{2}}$$

and

$$\bar{B}_P = \frac{3}{8} + \frac{3}{2}\frac{a_3\bar{\rho}}{a_1}$$

Note that the previous two equations can be solved for a_1 and a_3, giving

$$a_1 = 2\left(\frac{\pi}{\bar{\rho}}\right)^{\frac{1}{2}}\bar{A}_p,$$
$$a_3 = \frac{\pi^{\frac{1}{2}}}{2\bar{\rho}^{\frac{3}{2}}}\bar{A}_p\left(\frac{8}{3}\bar{B}_p - 1\right). \tag{2.6}$$

To use the relation $P(x) = \bar{P}(e^x - 1)$ requires that $e^x - 1$ be substituted for x in our expression for $P(x)$. Using $\bar{\rho} = e^{\rho} - 1$ we find that $(\bar{\rho} - x)$ should therefore be replaced by

$$\begin{aligned}\bar{\rho} - (e^x - 1) &= \bar{\rho} + 1 - e^{\rho-(\rho-x)}\\ &= \bar{\rho} + 1 - (\bar{\rho}+1)e^{-(\rho-x)}\\ &= (\bar{\rho}+1)[1 - e^{-(\rho-x)}]\\ &= (1+\bar{\rho})(\rho - x)[1 - \tfrac{1}{2}(\rho - x) \pm \cdots].\end{aligned}$$

In this way (2.4) yields

$$P(x) = -a_1(1+\bar{\rho})^{\frac{1}{2}}(\rho - x)^{\frac{1}{2}} + \left\{\frac{a_1}{4}(1+\bar{\rho})^{\frac{1}{2}} + a_3(1+\bar{\rho})^{\frac{3}{2}}\right\}(\rho - x)^{\frac{3}{2}} + \cdots$$

As before, (2.5) then gives

$$\frac{P_n}{n!}\rho^n n^{\frac{3}{2}} = A_p\left(1 + \frac{B_p}{n} + O\left(\frac{1}{n^2}\right)\right)$$

where

$$A_p = \frac{a_1(1+\bar{\rho})^{\frac{1}{2}}\rho^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}$$

and

$$B_p = \frac{3}{8}(1+\rho) + \frac{3a_3}{2a_1}(1+\bar{\rho})\rho.$$

Substituting for a_1 and a_3 as in (2.6) one finds

$$A_p = \left(\frac{(1+\bar{\rho})\rho}{\bar{\rho}}\right)^{\frac{1}{2}}\bar{A}_p \tag{2.7}$$

and

$$B_P = \tfrac{3}{8} + \{(1+\bar{\rho})\bar{B}_P - \tfrac{3}{8}\}\frac{\rho}{\bar{\rho}}.$$

The transformation of $\bar{T}(x)$ to $T(x)$ proceeds in the same way, although some of the constants are different due to the fact that the leading term for $T(x)$ is of order $(\rho-x)^{\frac{3}{2}}$ rather than $(\rho-x)^{\frac{1}{2}}$. The resulting relations are

$$A_T = \left(\frac{(1+\bar{\rho})\rho^{\frac{3}{2}}}{\bar{\rho}}\right)\bar{A}_T \tag{2.8}$$

and

$$B_T = \tfrac{15}{8} + \{(1+\bar{\rho})\bar{B}_T - \tfrac{15}{8}\}\frac{\rho}{\bar{\rho}}.$$

Here A_T, B_T are the constants such that

$$\frac{T_n}{n!}\rho^n n^{\frac{5}{2}} = A_T\left(1 + \frac{B_T}{n} + O\left(\frac{1}{n^2}\right)\right)$$

and $\bar{A}_T$, $\bar{B}_T$ play the same role for $\bar{T}_n$.

The transformation of $\bar{R}(x)$ to $R(x)$ is exactly like that of $\bar{P}(x)$ to $P(x)$ because the leading term for $R(x)$ is of the same order, $(\rho-x)^{\frac{1}{2}}$, as for $P(x)$. Therefore equations (2.7) apply directly with the subscript P replaced by R throughout.

The transformation of $\bar{S}(x)$ to $S(x)$ is different in certain constants because the leading term for $S(x)$ is of order $(\rho-x)^{-\frac{1}{2}}$. The relations for $S(x)$ which parallel (2.7) and (2.8) are

$$A_S = \left(\frac{1+\bar{\rho})\rho}{\bar{\rho}}\right)^{-\frac{1}{2}}\bar{A}_S \tag{2.9}$$

and

$$B_S = -\tfrac{1}{8} + \{(1+\bar{\rho})\bar{B}_S + \tfrac{1}{8}\}\frac{\rho}{\bar{\rho}}.$$

It is understood that

$$\frac{S_n}{n!}\rho^n n^{\frac{1}{2}} = A_S\left(1 + \frac{B_s}{n} + O\left(\frac{1}{n^2}\right)\right),$$

and similarly for $\bar{A}_S$ and $\bar{B}_S$.

Since $\mu_n = R_n/T_n$ and $\bar{\mu}_n = \bar{R}_n/\bar{T}_n$, we can apply (2.7) (as it applies to R_n instead of P_n) and (2.8), which give

$$\frac{\mu_n}{n} = A_\mu\left(1 + \frac{B_\mu}{n} + O\left(\frac{1}{n^2}\right)\right)$$

where

$$A_\mu = \left(\frac{(1+\bar\rho)\rho}{\bar\rho}\right)^{-1} \bar A_\mu \tag{2.10}$$

and

$$B_\mu = -\tfrac{3}{2} + \{(1+\bar\rho)\bar B_\mu + \tfrac{3}{2}\}\frac{\rho}{\bar\rho}.$$

Of course $\bar A_\mu$ and $\bar B_\mu$ are the constants giving the first two terms of the asymptotic expansion for $\bar\mu_n/n$.

To evaluate σ_n^2 asymptotically it is now assumed that $\bar A_S \bar A_T = \bar A_R^2$, which is equivalent to the asymptotic condition

$$\frac{\bar\sigma_n^2}{n} = \bar A_\nu + O\left(\frac{1}{n}\right).$$

One can then work from the relation

$$\sigma_n^2 = \frac{S_n}{T_n} + \mu_n - \mu_n^2$$

along with (2.8), (2.9), and (2.10) to calculate that

$$\frac{\sigma_n^2}{n} = A_\nu + O\left(\frac{1}{n}\right)$$

where

$$A_\nu = \left(\frac{(1+\bar\rho)\rho}{\bar\rho}\right)^{-1} \bar A_\nu + A_\mu^2\left(1-\frac{\rho}{\bar\rho}\right). \tag{2.11}$$

3. Case 1

The results of Section 2 can now be applied with $\bar P(x)$, $\bar T(x)$, $\bar R(x)$, and $\bar S(x)$ taken to be the counting series for Case 1 of [6 §2]. There $\bar\rho = 0.3102333374\ldots$, so we have $\rho = \ln(1+\bar\rho) = 0.2702052403$. In [6 §2] the values of $\bar A_p$, $\bar B_p$, $\bar A_T$, $\bar B_T$, $\bar A_R$, $\bar B_R$, $\bar A_S$, $\bar B_S$, $\bar A_\mu$, $\bar B_\mu$, and $\bar A_\nu$ are all determined to 10 significant digits. At the same time the various assumptions made in Section 2 concerning the series and their asymptotic behavior are verified. Consequently it is simply a matter of numerical calculation to use the values obtained in [6 §2] as input to equations (2.7), (2.8), (2.9), (2.10), and (2.11) to obtain:

$$A_p = 0.1707982096\ldots,$$
$$B_p = 0.3181314364\ldots,$$

where

$$P_n = n!\,\rho^{-n} n^{-\frac{3}{2}} A_P\left(1 + \frac{B_P}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Also

$$A_T = 0.0723245687\ldots,$$

$$B_T = 2.043867691\ldots,$$

where

$$T_n = n!\,\rho^{-n} n^{-\frac{5}{2}} A_T\left(1 + \frac{B_T}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Also

$$A_R = 0.0968670732\ldots,$$

$$B_R = 0.9170219933\ldots,$$

where

$$R_n = n!\,\rho^{-n} n^{-\frac{3}{2}} A_R\left(1 + \frac{B_R}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Also

$$A_\mu = 1.339339038\ \ldots,$$

$$B_\mu = -1.117847390\ldots,$$

where

$$\mu_n = n\left(A_\mu + \frac{B_\mu}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Also

$$A_S = 0.1297377923\ldots,$$

$$B_S = -0.6559227561\ldots,$$

where

$$S_n = n!\,\rho^{-n} n^{-\frac{1}{2}} A_S\left(1 + \frac{B_S}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Finally

$$A_v = 0.5229767183\ldots,$$

where

$$\sigma_n^2 = nA_v + O(1).$$

4. Case 2

Once again the results of section 2 can be applied with $\bar{P}(x)$, $\bar{T}(x)$, $\bar{R}(x)$, and $\bar{S}(x)$ taken to be the counting series for Case 2 of [6 §3]. There $\bar{\rho} = 0.2367771669\ldots$, so we have $\rho = \ln(1+\bar{\rho}) = 0.212508937\ldots$. In [6 §3] the values of $\bar{A}_P$, $\bar{B}_P$, $\bar{A}_T$, $\bar{B}_T$, $\bar{A}_R$, $\bar{B}_R$, $\bar{A}_S$, $\bar{B}_S$, $\bar{A}_\mu$, $\bar{B}_\mu$, and $\bar{A}_\nu$ are all determined to 10 significant digits. As in the previous section, the assumptions concerning the series and their asymptotic behavior are verified. Hence we used the values obtained in [6 §3] as input to equations (2.7), (2.8), (2.9), (2.10), and (2.11) to obtain:

$$P_n = n!\,\rho^{-n} n^{-\frac{3}{2}} A_P\left(1 + \frac{B_P}{n} + O\left(\frac{1}{n^2}\right)\right)$$

where

$$A_P = 0.1928169230\ldots,$$
$$B_P = 0.5090632210\ldots,$$

Also

$$T_n = n!\,\rho^{-n} n^{-\frac{5}{2}} A_T\left(1 + \frac{B_T}{n!} + O\left(\frac{1}{n^2}\right)\right)$$

where

$$A_T = 0.0794186525\ldots,$$
$$B_T = 2.259683002\ldots,$$

Also

$$R_n = n!\,\rho^{-n} n^{-\frac{3}{2}} A_R\left(1 + \frac{B_R}{n} + O\left(\frac{1}{n^2}\right)\right)$$

where

$$A_R = 0.1093548242\ldots,$$
$$B_R = 1.047014483\ldots,$$

Also

$$\mu_n = n\left(A_\mu + \frac{B_\mu}{n} + O\left(\frac{1}{n^2}\right)\right)$$

where

$$A_\mu = 1.376941321\ldots,$$
$$B_\mu = -1.212668519\ldots,$$

Also

$$S_n = n!\,\rho^{-n} n^{-\frac{1}{2}} A_S\left(1 + \frac{B_S}{n} + O\left(\frac{1}{n^2}\right)\right)$$

where

$$A_S = 0.1505751762\ldots,$$

$$B_S = -0.5660531466\ldots,$$

Finally

$$\sigma_n^2 = nA_v + O(1)$$

where

$$A_v = 0.3670613002\ldots.$$

5. Summary and conclusions

In this paper we have completed our analysis of the enumeration of the twelve classes of phylogenies. The two final cases which we studied here both allow multiple labelling of vertices. They are: (i) no vertices of degree two and interior vertices may possibly be labelled, and (ii) there are no restrictions on vertex degree but only vertices of degree one or two can be labelled. The exact and asymptotic number of phylogenies with given magnitude, along with the mean and variance of their order was also determined.

The method used to derive cases 1 and 2 from the previously analyzed singleton-label cases is a general one, which could have been applied to the other multiple-label classes of phylogenetic trees analysed in previous papers by the authors. It is in the nature of the general transformation that a linearly growing variance of the order in a singleton-label case implies the same behavior of the variance of the orde1 in the corresponding multiple-label case. In fact the variance of the order is indeed linear in each class studied.

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TRIAD COUNT STATISTICS

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The triad counts of a graph G are the numbers of various distinct induced subgraphs of order 3. If G is an undirected graph, there are 4 triad counts, and if G is a digraph, there are 16 triad counts. A multigraph is a sequence $G=(G_1,\ldots,G_r)$ of graphs and digraphs defined on a common vertex set. The concept of triad counts is generalized to multigraphs with colored vertices, edges, and arcs. It is shown how triad counts in multigraphs can be used in various kinds of statistical analyses of graph data. In particular, probability distributions are investigated of the triad counts in random multigraphs.

1. Introduction

A dyad of a graph of digraph is an induced subgraph of order 2, and a triad is an induced subgraph of order 3. For a graph or digraph G of order n there are $\binom{n}{2}$ dyads and $\binom{n}{3}$ triads that can be partitioned into different equivalence classes according to isomorphism. If G is a graph, there are at most 4 nonisomorphic triads (Fig. 1), and if G is a digraph, there are at most 16 nonisomorphic triads (Fig. 2). The frequencies of isomorphic dyads and triads in G are called the dyad and triad counts of G.

In statistical applications of random graph models the dyad and triad counts have been found useful both as summary statistics for exploratory analyses and as sufficient statistics for inference on particular random graph models. Some further comments and references on the use of dyad and triad counts in statistics are given in the next section. This shows that there is a need for extending the concept of dyad and triad counts to multigraphs $G=(G_1,\ldots,G_r)$, that is to ordered sequences of graphs and digraphs on a common vertex set. Various other extensions such as signed graphs, valued graphs and colored graphs are also discussed. Section 3 gives formulae for the numbers of nonisomorphic dyads and triads that exist for colored multigraphs. Section 4 discusses properties of the triad counts and some open extremal graph problems of interest in this connection. Section 5 gives some probabilistic properties of the triad counts for a random graph model that can be used for colored multigraphs.

2. Statistical applications

Data to be modelled by random graphs can be obtained from social contact networks, paired comparison experiments, political dominance patterns, unreli-

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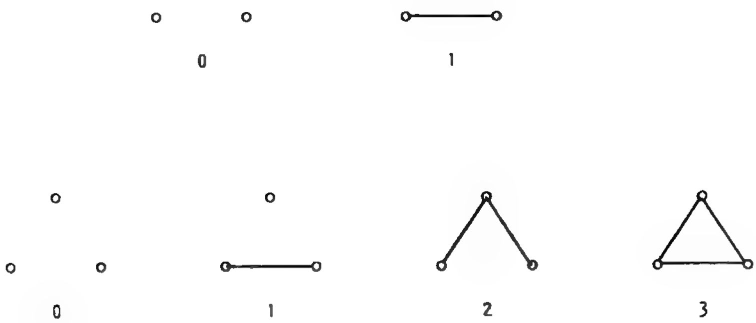

Fig. 1.

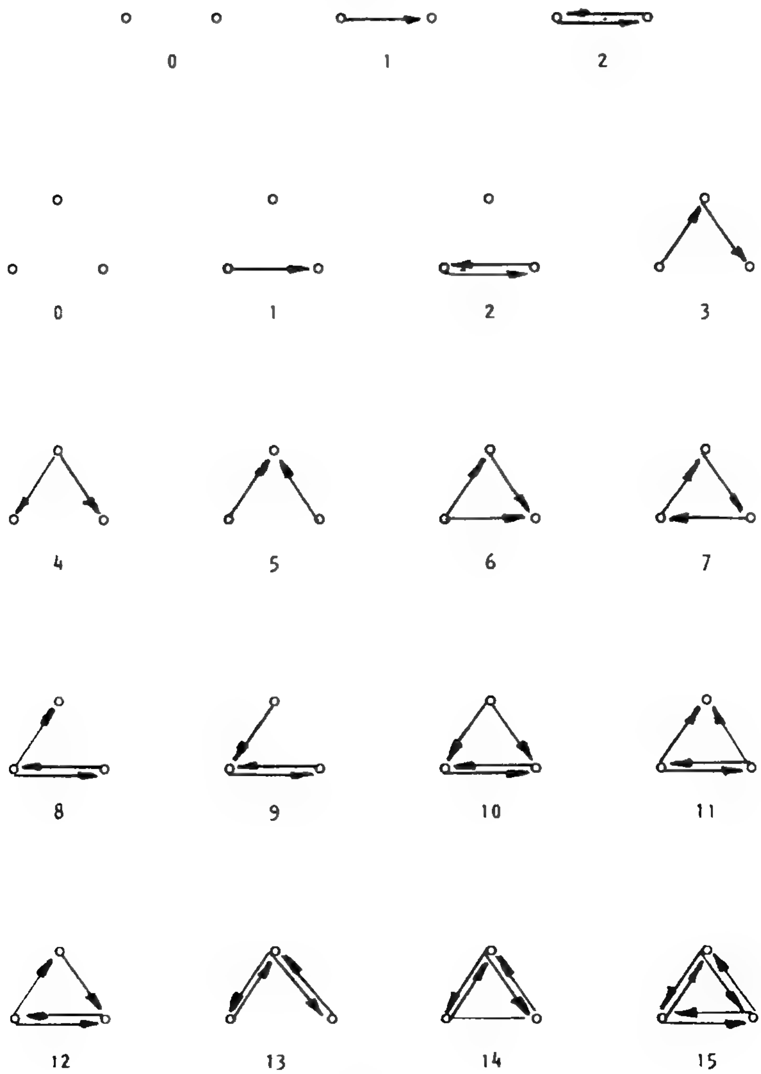

Fig. 2.

able component systems, and so forth. Often there is more than a single binary relationship to be investigated, and often there are variables defined both on the objects and on the pairs of objects involved. For instance, the vertices of a graph model can be people with vertex variables gender, age, income, etc. and edges and arcs for contacts having contact variables duration, purposes, gain, etc. Exploratory statistical analyses of graph data can be based on summary statistics like the triad counts. If different values on vertex, edge, and arc variables are represented by colors we can speak of triad counts in colored multigraphs. For instance, 4 arc colors are needed to represent the occurrences and non-occurrences of two different kinds of relationship, and 3 arc colors are needed to represent a single relationship that might be strong, weak, or absent.

Several authors have used triad counts to investigate structural properties of empirical networks and to estimate and test particular random graph models. The 16 triad counts in digraphs have been extensively studied in social science applications. See for instance Holland and Leinhardt [15, 16] and other articles in the same proceedings volumes. Transitivity indices defined in terms of triad counts were considered by Frank [6], Frank and Harary [10] and Holland and Leinhardt [13, 14, 15]. Frank [3, 4] used triad counts to study transitivity and clustering properties of graphs. Frank and Harary [8] used triad counts in signed graphs to investigate structural balance. Frank and Strauss [11] proved that the triad counts are sufficient statistics for a particular class of Markov random graph models.

We can distinguish between two different main uses of triad counts in statistical analysis. The triad counts can be used for estimating and testing particular random graph models, for instance estimating a Markov model or testing a pure randomness model versus a Markov model. A pure randomness model can here be a Bernoulli graph or a dyad independence model of the type that is described in Section 5.

Triad counts can also be used in exploring and modelling graph data, for instance by using normalized triad counts as regressors for predicting graph properties or as discriminators for graph classification. By normalized triad counts is here meant observed triad counts divided by their estimated expected values according to some random graph model. Several examples of these uses can be found in the references given above.

3. Numbers of dyads and triads

The appropriate choice and scaling of variables for multivariate graph data can be partly guided by considering the effects on the numbers of dyads and triads, that is on the number of statistics to be used in the subsequent analysis. In order to determine these numbers in general we introduce a multigraph with colored vertices, edges and arcs.

Theorem 1. *For multigraphs with a vertex colors, b edge colors and c arc colors there are* $\binom{ac+1}{2}b$ *nonisomorphic dyads and*

$$\binom{abc^2+2}{3} - a^2b^2c^2\binom{c}{2}$$

nonisomorphic triads.

Proof. Burnside's lemma (see for instance Harary [12], p. 181) can be applied to count the numbers of nonisomorphic dyads and triads. The number of nonisomor-

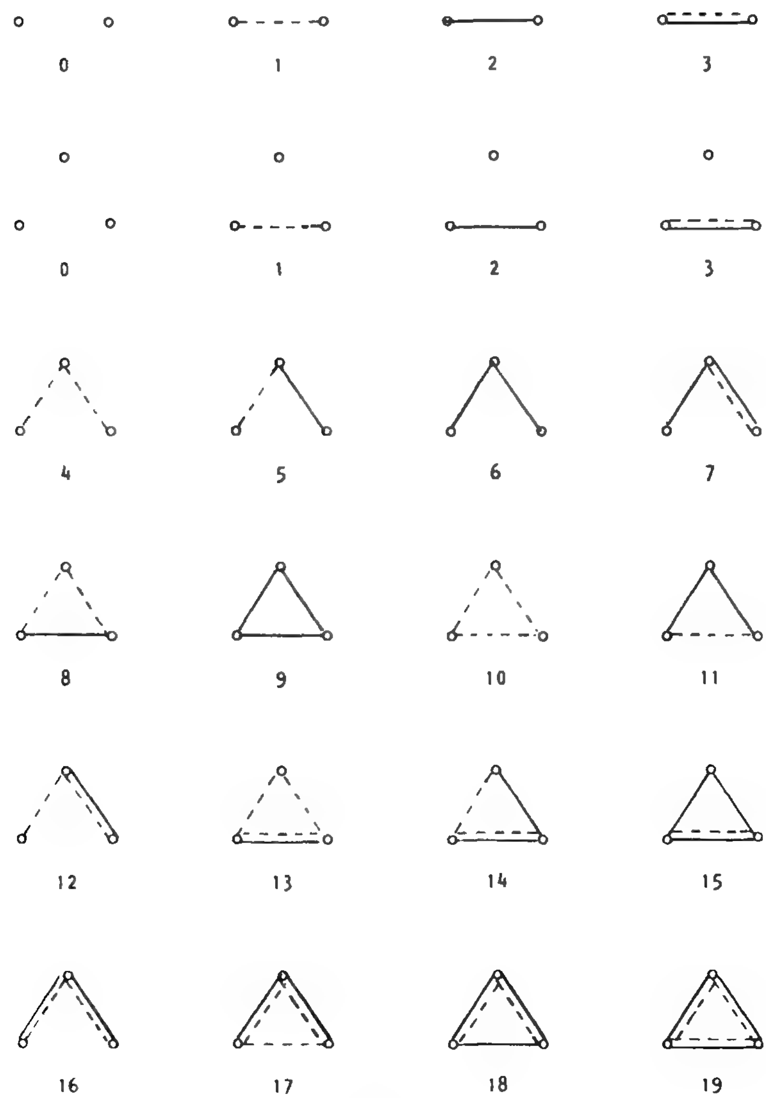

Fig. 3.

phic dyads is given by the average number of fixpoints of the vertex permutations applied to all a^2bc^2 labeled dyads that can be formed with the colors. For the identity permutation all labeled dyads are fix-points, and for the switch permutation there are abc fixpoints. Thus the number of nonisomorphic dyads is given by $\frac{1}{2}(a^2bc^2 + abc) = \binom{ac+1}{2}b$. For triads there are six vertex permutations: the identity, three switches, and two rotations. The numbers of fixpoints are found to be $a^3b^3c^6$ for the identity, $a^2b^2c^3$ for any switch, and abc^2 for any rotation. Thus the number of nonisomorphic triads is given by the average number of fixpoints

$$\frac{(a^3b^3c^6 + 3a^2b^2c^3 + 2abc^2)}{6} = \binom{abc^2+2}{3} - a^2b^2c^2\binom{c}{2}. \quad \square$$

By putting $c = 1$ and $b = 1$, respectively, we obtain the following two corollaries.

Corollary 1.1. *For graphs with a vertex colors and b edge colors there are* $\binom{a+1}{2}b$ *distinct dyads and* $\binom{ab+2}{3}$ *distinct triads.*

Corollary 1.2. *For digraphs with a vertex colors and c arc colors there are* $\binom{ac+1}{2}$ *distinct dyads and* $\binom{ac^2+2}{3} - a^2c^2\binom{c}{2}$ *distinct triads.*

By putting $a = 1$, $b = 2^g$ and $c = 2^d$ we obtain the following corollary.

Corollary 1.3. *For multigraphs consisting of g graphs and d digraphs there are* $(2^d + 1)2^{d+g-1}$ *distinct dyads and* $\binom{2^{g+2d}+1}{3} + 2^d\binom{2^{g+d}+1}{2}$ *distinct triads.*

In particular, there are 20 triads (Fig. 3) for multigraphs consisting of two graphs, and there are 104 triads for multigraphs consisting of a graph and a digraph.

4. Dyad and triad counts

Consider colored multigraphs with a vertex colors, b edge colors and c arc colors. Denote the dyad counts of a colored multigraph G by $(r_0, \ldots, r_k)$, where $k+1$ is the number of nonisomorphic dyads, and these dyads have been labeled according to some fixed order. The triad counts of G are denoted by $(t_0, \ldots, t_l)$, where $l+1$ is the number of nonisomorphic triads also labeled according to some fixed order. If G has order n, then

$$\sum_{i=0}^{k} r_i = \binom{n}{2}, \qquad \sum_{j=0}^{l} t_j = \binom{n}{3}.$$

The dyad counts can be obtained from the triad counts. In fact, if r_{ij} is the number of i-dyads in the j-triad, then

$$\sum_{j=0}^{l} r_{ij}t_j = (n-2)r_i, \quad i = 0, \dots, k.$$

Various properties of G can be expressed in terms of these counts. For instance, for a single graph G take r_i and t_j as the numbers of dyads of size i for $i = 0, 1$ and triads of size j for $j = 0, 1, 2, 3$. If G has order n, size r, degrees $d_1, \dots, d_n$ and s two-paths, then by using that

$$s = t_2 + 3t_3 = \sum_{i=1}^{n} \binom{d_i}{2}$$

we easily find that the mean and variance of the degrees can be obtained from the triad counts according to

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{2(t_1 + 2t_2 + 3t_3)}{n(n-2)},$$

$$\frac{1}{n}\sum_{i=1}^{n} (d_i - \bar{d})^2 = \bar{d}(1-\bar{d}) + \frac{2(t_2 + 3t_3)}{n}.$$

A problem of general interest to statistical applications is to find upper and lower bounds to the triad counts of colored multigraphs G with specified dyad counts. For instance we easily find that $t_3 \leq (n-2)\frac{1}{3}r$ for graphs of order n and size r. Can better bounds be found if the graph has special properties like being planar, connected, transitive, and so forth? Frank [7] considered triads in planar graphs. For instance, it holds that $t_3 \leq (n-2)^2$ for planar graphs of order n. For digraphs of odd order n, there cannot be more than $\frac{1}{4}\binom{n+1}{3}$ 3-cycles. Frank and Harary [9] considered maximum triad counts in graphs and digraphs of order n, and Bollobás [1] has a lot of material on related issues. For instance, for graphs of order n and size r

$$\max t_3 = \binom{v}{3} + \binom{r - \binom{v}{2}}{2}$$

where v is given by $\binom{v}{2} \leq r < \binom{v+1}{2}$. A lower bound to t_3 for $r \geq \frac{1}{4}n^2$ is given by $r(4r - n^2)/3n$, and a lower bound to t_3 for $\frac{1}{4}n^2 \leq r \leq \frac{1}{3}n^2$ is given by $n(4r - n^2)/9$ (Bollobás [1], Corollary 1.6 p. 297 and Corollary 1.9 p. 301). For $n \to \infty$ and $r/\binom{n}{2} \to p$ where $\frac{1}{2} \leq p \leq \frac{2}{3}$ we obtain that

$$2(2p-1)/3 \leq t_3 \Big/ \binom{n}{3} \leq p^{3/2}.$$

A limiting version of interest of the problem for colored multigraphs is to let $n \to \infty$ so that the relative dyad counts tend to specified limits $r_i/\binom{n}{2} \to p_i$ for $i = 0, \dots, k$ and ask for bounds to the relative triad counts $t_j/\binom{n}{3}$ for $j = 0, \dots, l$.

5. Random colored multigraphs

In random graph theory the most common models are the uniform random graph of fixed size and the Bernoulli graph on a fixed finite vertex set, say $[n]=\{1,\ldots,n\}$. A uniform random graph of size r and order n is a random graph that assigns the same probability $\binom{n}{r}^{-1}$ to each graph of size r on $[n]$. A Bernoulli graph of order n with edge probability p is a random graph that assigns probability

$$p^r(1-p)^{\binom{n}{2}-r}$$

to each graph of size r on $[n]$ for $r=0,\ldots,\binom{n}{2}$. These models are treated extensively in the recent monographs by Bollobás [2] and Palmer [17].

A natural extension of the Bernoulli-graph to digraphs is the dyad independence model considered by Frank [6]. According to this model the $\binom{n}{2}$ random dyads are independent identically distributed with a probability distribution that is invariant under isomorphism. Thus, the random digraph distribution is specified by the probabilities assigned to the dyads of size i for $i=0$, 1, 2.

For a colored multigraph with no vertex colors we can define a similar general dyad independence model by specifying the probabilities p_i assigned to the nonisomorphic dyads $i=0,\ldots,k$. Here $p_i \geq 0$ and $p_0+\cdots+p_k=1$. Each labeled colored multigraph on $[n]$ with dyad counts $(r_0,\ldots,r_k)$ is assigned a probability

$$\prod_{i=0}^{k} p_i^{r_i}.$$

Let $(R_0,\ldots,R_k)$ be the random dyad counts and $(T_0,\ldots,T_l)$ the random triad counts of a general dyad independence model. Now $(R_1,\ldots,R_k)$ is multinomially distributed with parameters $\binom{n}{2}$ and $(p_1,\ldots,p_k)$. The distribution of the triad counts is not easy to determine. However, the expected values and the variances and covariances can be obtained by generalizing the methods used by Frank [5]. We have the following result.

Theorem 2. *For a random colored multigraph of order n the triad counts $(T_0,\ldots,T_l)$ have expected values $ET_i=\binom{n}{3}P_i$ and covariances*

$$Cov(T_i,T_j)=\binom{n}{3}P_i(\delta_{ij}-P_j)+12\binom{n}{4}(P_{ij}-P_iP_j)$$

where P_i is the probability that a random triad is isomorphic to the i-triad, P_{ij} is the probability that two random triads with two vertices in common are isomorphic to the i-triad and the j-triad, resp., and δ_{ij} is 1 or 0 according to whether or not $i=j$.

In order to determine P_i it is convenient to use that

$$P_i=\frac{3!}{r_{0i}!\cdots r_{ki}!}p_0^{r_{0i}}\cdots p_k^{r_{ki}}w_i$$

Table 1. Triad probabilities of digraphs.
(Dyads and triads labeled according to Fig. 2.)

Triad i	Dyad counts $r_{0i}r_{1i}r_{2i}$	Labelings	Proportion w_i	Probability P_i
0	300	1	1	p_0^3
1	210	6	1	$3p_0^2p_1$
2	201	3	1	$3p_0^2p_2$
3	120	6	$\frac{1}{2}$	$3p_0p_1^2/2$
4	120	3	$\frac{1}{4}$	$3p_0p_1^2/4$
5	120	3	$\frac{1}{4}$	$3p_0p_1^2/4$
6	030	6	$\frac{3}{4}$	$3p_1^3/4$
7	030	2	$\frac{1}{4}$	$p_1^3/4$
8	111	6	$\frac{1}{2}$	$3p_0p_1p_2$
9	111	6	$\frac{1}{2}$	$3p_0p_1p_2$
10	021	3	$\frac{1}{4}$	$3p_1^2p_2/4$
11	021	3	$\frac{1}{4}$	$3p_1^2p_2/4$
12	021	6	$\frac{1}{2}$	$3p_1^2p_2/2$
13	102	3	1	$3p_0p_2^2$
14	012	6	1	$3p_1p_2^2$
15	003	1	1	p_2^3

Table 2. Triad probabilities of multigraphs consisting of two graphs. (Dyads and triads labeled according to Fig. 3.)

Triad i	Dyad counts $r_{0i}r_{1i}r_{2i}r_{3i}$	Labelings	Proportion w_i	Probability P_i
0	3000	1	1	p_0^3
1	2100	3	1	$3p_0^2p_1$
2	2010	3	1	$3p_0^2p_2$
3	2001	3	1	$3p_0^2p_3$
4	1200	3	1	$3p_0p_1^2$
5	1110	6	1	$6p_0p_1p_2$
6	1020	3	1	$3p_0p_2^2$
7	1011	6	1	$6p_0p_2p_3$
8	0210	3	1	$3p_1^2p_2$
9	0030	1	1	p_2^3
10	0300	1	1	p_1^3
11	0120	3	1	$3p_1p_2^2$
12	1101	6	1	$6p_0p_1p_3$
13	0201	3	1	$3p_1^2p_3$
14	0111	6	1	$6p_1p_2p_3$
15	0021	3	1	$3p_2^2p_3$
16	1002	3	1	$3p_0p_3^2$
17	0102	3	1	$3p_1p_3^2$
18	0012	3	1	$3p_2p_3^2$
19	0003	1	1	p_3^3

where $(r_{0i}, \ldots, r_{ki})$ are the dyad counts of the i-triad, and w_i is the proportion of triads isomorphic to the i-triad among the triads which have the same dyad counts as the i-triad. In particular, $w_i = 1$ if the i-triad is uniquely specified by its dyad counts. Otherwise there are several non-isomorphic triads which have the same dyad counts as the i-triad. For each such triad we find how many isomorphic labelings there are and then obtain w_i as the proportion of labelings for the i-triad.

Consider for example a single digraph. Dyads and triads are given in Fig. 2. The dyad counts of the triads, the numbers of isomorphic labelings, and the probabilities are given in Table 1. Table 2 gives corresponding quantities for a multigraph consisting of two graphs. We note that when $c = 1$ all nonisomorphic triads are uniquely specified by their dyad counts.

The main use of Theorem 2 in exploratory statistics is that it shows how the triad counts can be normalized to define regressors and discriminators. We normalize according to $X_i = T_i/\binom{n}{3}\hat{P}_i$ where $\hat{P}_i$ is obtained from P_i by substituting maximum likelihood estimators for the dyad probabilities, that is p_j is replaced by $\hat{p}_j = R_j/\binom{n}{2}$ for $j = 0, \ldots, k$. The probabilistic properties of $(X_1, \ldots, X_l)$ are not well known.

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SCORE SEQUENCES: LEXICOGRAPHIC ENUMERATION AND TOURNAMENT CONSTRUCTION

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In this paper we give an algorithm for generating all tournament score sequences of a given length in lexicographic order. We also show how to construct a tournament with a given score sequence.

1. Introduction

A *tournament of order n* is a loopless digraph $T = \langle V, A\rangle$, where $V = \{1, 2, \ldots, n\}$, such that there exists exactly one arc between any pair of distinct vertices i and j. The *score of a vertex i*, denoted by $s_T(i)$ (or simply $s(i)$), is defined as the outdegree of i. Unless otherwise specified, we shall assume that $s(1) \leqslant s(2) \leqslant \cdots \leqslant s(n)$. The sequence $\langle s(i)\rangle = \langle s(1), s(2), \ldots, s(n)\rangle$ is called the *score sequence of T*.

To be able to enumerate all score sequences of a given length, one must first know a necessary and sufficient condition for a non-decreasing sequence of integers to be a score sequence. Such a condition is given in a characterization theorem proved by Landau [4] in 1953.

Theorem 1 [4]. *A non-decreasing sequence of n integers $\langle s(i)\rangle$ is a score sequence if and only if*

$$\sum_{i=1}^{k} s(i) \geqslant \binom{k}{2}$$

for each $k = 1, 2, \ldots, n$ with equality for $k = n$.

We shall find an equivalent characterization of score sequences which is very helpful in the lexicographic enumeration algorithm established in this paper.

Let $\langle s(i)\rangle$ be any sequence of integers. We define the (*transitive*) *deviation sequence of* $\langle s(i)\rangle$ to be the sequence $\langle d(i)\rangle = \langle s(i) - i + 1\rangle$. We shall call $d(i)$ the *deviation of* $s(i)$. It is easy to see that $\langle s(i)\rangle$ is non-decreasing if and only if $d(i) - d(i+1) \leqslant 1$ for each $i < n$. It is also easy to check that for each $k = 1, 2, \ldots, n$, $\sum_{i=1}^{k} d(i) = \sum_{i=1}^{k} s(i) - \binom{k}{2}$. From this, we see that a non-decreasing sequence of n integers $\langle s(i)\rangle$ is a score sequence if and only if its

deviation sequence $\langle d(i)\rangle$ satisfies $\sum_{i=1}^{k} d(i) \geq 0$ for $k=1, 2, \ldots, n$ with equality for $k=n$.

2. Lexicographic enumeration of score sequences

Let $\langle s(i)\rangle$ and $\langle s'(i)\rangle$ be score sequences of length n. We say that $\langle s(i)\rangle$ *precedes* $\langle s'(i)\rangle$ if there exists a positive integer $k \leq n$ such that $s(i)=s'(i)$ for each $1 \leq i < k$ and $s(k) < s'(k)$. (They are *equal* if equality holds for all i.) In symbols we shall write $\langle s(i)\rangle \leq \langle s'(i)\rangle$ if $\langle s(i)\rangle$ precedes $\langle s'(i)\rangle$. We say that $\langle s'(i)\rangle$ is the *successor* of $\langle s(i)\rangle$ if they are distinct, $\langle s(i)\rangle \leq \langle s'(i)\rangle$, and $\langle s'(i)\rangle \leq \langle s''(i)\rangle$ whenever $\langle s(i)\rangle \leq \langle s''(i)\rangle$.

An enumeration of all score sequences of a given length with the property that the successor of any score sequence follows it immediately in the list is called a *lexicographic enumeration*.

Let us take note of two important facts. First, the score sequence $\langle 0, 1, 2, \ldots, n-1\rangle$ is not the successor of any score sequence of length n and hence it is always the first in the lexicographic enumeration. Second, $\langle s(i)\rangle$ has no successor if and only if $s(n)-s(1) \leq 1$.

If we know the first sequence in a lexicographic enumeration, then we can complete the work provided we know how to get the successor of any given sequence.

The successor $\langle s'(i)\rangle$ of $\langle s(i)\rangle$, if there is any, can be obtained as follows.

(1) Determine the maximum k such that $s(n)-s(k) \geq 2$.

(2) Let $s'(i)=s(i)$ for all i, $i<k$.

(3) Let $s'(k)=s(k)+1$.

(4) Let $s'(j)=s(k)+1$ until $\sum_{i=1}^{j} s'(i) < \binom{j}{2}$.

(5) Let t be the minimum j such that $\sum_{i=1}^{j} s'(i) < \binom{j}{2}$;
Set $s'(t)=\binom{t}{2}-\sum_{i=1}^{t-1} s'(i)$.

(6) Let $s'(i)=i-1$ for all i, $t<i \leq n$.

3. Constructing a tournament with a given score sequence

One method of constructing a tournament with a given score sequence can be found in [1]. Here we shall give another method of construction and prove its validity.

Lemma 1. *Let $\langle s(i)\rangle$ be a score sequence of length n with deviation sequence $\langle d(i)\rangle$.*

(a) *If $\max\{d(i)\}=M>0$, then for each $1 \leq k \leq M$, there exists a vertex x such that $d(x)=k$.*

(b) *If $\min\{d(i)\}=m<0$, then for each $-1 \geq k \geq m$, there exists a vertex x such that $d(x)=k$.*

Proof of (a). Certainly, (a) is true for $k = M$. We shall prove that (a) is true for all $1 \leqslant k \leqslant M$ by (backwards) induction on k. Let $1 \leqslant k < M$ and assume that (a) holds for $k+1$, i.e. there exists a vertex x such that $d(x) = k+1$. Since $d(n) \leqslant 0$, there exists $t > x$ such that $d(x) = d(t) > d(t+1)$. Since $d(t) \leqslant d(t+1)+1$, we have $d(t+1) = d(x) - 1 = k$. Hence (a) holds. Part (b) can be proven using a similar kind of argument. □

Lemma 2. *Let $\langle s(i) \rangle$ be a score sequence of length n with deviation sequence $\langle d(i) \rangle$. If c is the number of negative terms, then $c \geqslant \max\{d(i)\}$.*

Proof. Let $p = \max\{d(i)\rangle$. If $p = 0$, then $c = 0$ by Theorem 2 and the Lemma holds. If $p > 0$, we consider the following cases.

Case 1. *There exists a vertex k such that $d(k) < 0$ and $|d(k)| \geqslant p$.*

By Lemma 1, $c \geqslant |d(k)| \geqslant p$.

Case 2. *For each negative deviation $d(k)$, $|d(k)| < p$.*

Let $q = \max\{|d(i)\}| : d(i) < 0\}$ and suppose that $c < p$. Then using Lemma 1, $\sum_{d(i)<0} |d(i)| \leqslant 1 + 2 + \cdots + q + (p-q)q$. But by Theorem 1, $\sum_{d(i)<0} |d(i)| = \sum_{d(i)>0} d(i)$ and by Lemma 1, $\sum_{d(i)>0} d(i) \geqslant 1 + 2 + \cdots + p$. Hence, $q(q+1)/2 + (p-q)q > p(p+1)/2$. This gives the quadratic inequality $p^2 - (2q-1)p + q(q-1) < 0$ which implies that $q - 1 < p < q$. This is absurd since p and q are integers. Hence, $c \geqslant p$. □

We are now prepared to describe and validate an algorithm for constructing a tournament with a given score sequence, which is a main result of this paper.

Construction algorithm. Let $\langle s(i) \rangle$ be a score sequence of length n with deviation sequence $\langle d(i) \rangle$. First we take n vertices arranged horizontally and labelled $1, 2, \ldots, n$ from left to right.

Step 1. Subdivide $\langle d(i) \rangle$ into maximal non-increasing segments and denote by p the number of segments in the subdivision. Let n_i be the number of negative deviations in the ith segment (counting from left to right).

Step 2. Let j be the least integer such that $d(j) > 0$. If no such j exists, go to Step 6. Else, determine the least integer q such that $\sum_{i \leqslant q} n_i \geqslant d(j)$. For each i in the segments to the left of the qth segment such that $d(i) < 0$, let $d'(i) = d(i) + 1$ and draw the arc ji.

Step 3. Let $\sigma = d(j) - \sum_{i<q} n_i$ and choose σ smallest (negatively largest) devia-

tions $d(i)$ in the qth segment. For each such $d(i)$, let $d'(i) = d(i) + 1$ and draw the arc ji. Let $d'(j) = 0$.

Step 4. For all other deviations $d(i)$ not changed in the preceding steps, let $d'(i) = d(i)$.

Step 5. If $\langle d'(i)\rangle \neq \langle 0\rangle$, go to Step 1 using $\langle d'(i)\rangle$ in place of $\langle d(i)\rangle$.

Step 6. Whenever $u < v$ and there is no arc between u and v, draw the arc vu.

Step 7. The resulting digraph is a tournament with score sequence $\langle d(i)\rangle$.

Let us now analyze Algorithm 3 to see its validity. Clearly, Step 1 can always be carried out. Step 2 can be done in view of Lemma 2. Step 3 can be implemented because of Step 2. We can obviously do Step 4. Furthermore, we see that after this step, $\langle d'(i)\rangle$ satisfies $d'(i) - d'(i+1) \leqslant 1$ for all $1 \leqslant i < n$. Now, what happens when we reach the stage $\langle d'(i)\rangle = \langle 0\rangle$? Let G' be the digraph formed at this stage. Then it is easily seen that for each vertex i in G', we have

$$od_{G'}(i) = \begin{cases} d(i), & \text{if } d(i) \geqslant 0 \\ 0, & \text{if } d(i) < 0 \end{cases}$$

$$id_{G'}(i) = \begin{cases} 0, & \text{if } d(i) \geqslant 0 \\ -d(i), & \text{if } d(i) < 0 \end{cases}$$

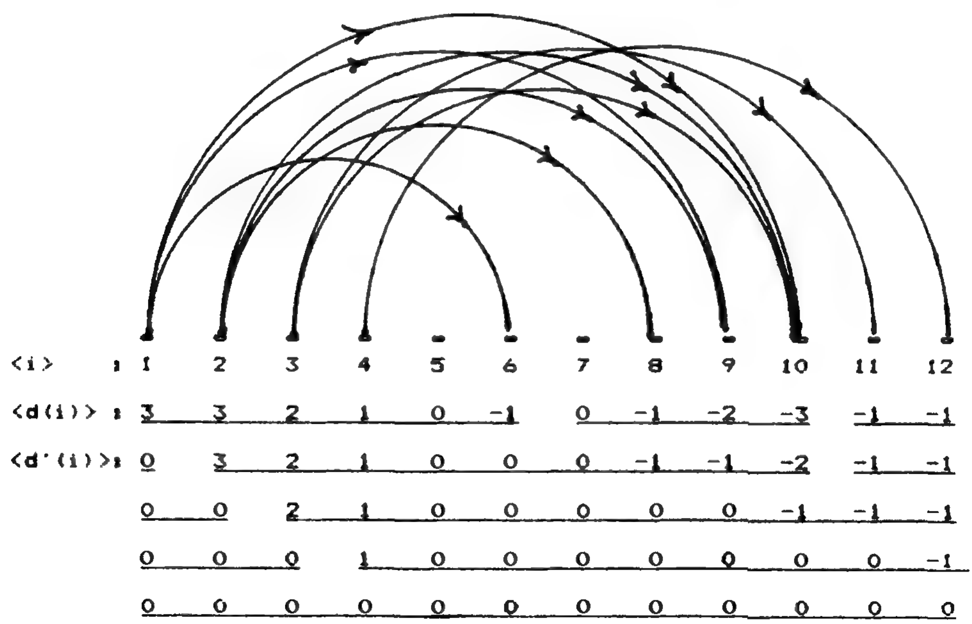

Fig. 1. The above digraph is the result after using Algorithm 3 up to the stage where $\langle d'(i)\rangle = \langle 0\rangle$. We complete it into a tournament by adding all the arcs ij, $(i > j)$, between two vertices i and j which are not yet jointed by any arc.

Let T be the tournament formed after Step 6, and let i be any vertex of T. If $d(i) \geq 0$, then we have $s(i) = od(i) = od_{G'}(i) + i - 1 = s(i)$. If $d(i) < 0$, then we have $id(i) = id_{G'}(i) + n - i = -d(i) + n - i$ and hence $s(i) = od(i) = (n-1) - id(i) = d(i) + i - 1 = s(i)$. Therefore, the score sequence of T is $\langle s(i) \rangle$.

Example. Construct a tournament with score sequence $\langle s(i) \rangle = \langle 3, 4, 4, 4, 4, 4, 6, 6, 6, 6, 9, 10 \rangle$.

We have $\langle d(i) \rangle = \langle 3, 3, 2, 1, 0, -1, 0, -1, -2, -3, -1, -1 \rangle$. Using the algorithm up to the point when $\langle d'(i) \rangle = \langle 0 \rangle$, we obtain the graph G' shown in Fig. 1. (The maximal non-increasing segments of each $\langle d'(i) \rangle$ are underlined.)

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PROBLEMS ON CHAIN PARTITIONS

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Problems on chain partitions

Recently I have come across several fundamental problems concerning the existence of partitions of posets into chains satisfying certain conditions. Other problems of this sort have been open and frustrating researchers for years. I have collected these problems here, together with a few of my own, to give them wider exposure. The problems can be easily stated for a broad mathematical audience. It is likely that their solutions require techniques and insights which differ greatly from those normally employed in the contexts in which the problems were initially developed.

The presentation of the problems follows a review of the necessary notation and terminology. We first discuss problems about the poset of subsets of a finite set, and then we give problems for other families of posets.

Notation and terminology

Throughout we consider finite posets $P = (P, \leqslant)$. A totally ordered subset of P, say $C = \{x_1, \ldots, x_r\} \subseteq P$ with $x_1 < x_2 < \cdots < x_r$, is called a *chain*. Such a chain is *saturated* (also called unrefinable or consecutive) if for all $i \geqslant 2$, x_{i+1} *covers* x_i, that is, $x_i \leqslant y < x_{i+1}$ for y in P only if $y = x_i$. An *antichain* is a totally unordered subset of P. The *width* of P, denoted $d_1(P)$, is the maximum size of an antichain in P. Let A be a maximum-sized antichain, and let $\boldsymbol{C} = \{C_1, \ldots, C_s\}$ be a partition of P into chains C_i. Since any chain C_i intersects the antichain A at most once, it follows that

$$|\boldsymbol{C}| = s \geqslant |A| = d_1(P).$$

A theorem of Dilworth [5] states that this bound on the number of chains in a chain partition is best-possible: There exists $\boldsymbol{C}$ such that $|\boldsymbol{C}| = d_1(P)$.

We next recall an important generalization of Dilworth's Theorem. Given $k \geqslant 1$, a subset F of P is a *k-family* if it can be expressed as the union of at most k

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antichains (or, equivalently, if $|F \cap C| \leq k$ for every chain $C \subseteq P$). Let $d_k(P) = \max\{|F| : F \subseteq P \text{ a } k\text{-family}\}$. A chain partition $\boldsymbol{C} = \{C_1, \ldots, C_s\}$ induces an upper bound on the size of a k-family F in the following way:

$$|F| = \sum_i |F \cap C_i| \leq \sum_i \min(k, |C_i|).$$

We denote this last sum by $d(k, \boldsymbol{C})$. By taking F to be a maximum-sized k-family, we obtain $d(k, \boldsymbol{C}) \geq d_k(P)$. If $d(k, \boldsymbol{C}) = d_k(P)$, $\boldsymbol{C}$ is said to be k-saturated. The Greene–Kleitman theorem [8] states that for all k and P there exists a k-saturated chain partition $\boldsymbol{C}$. Dilworth's Theorem is the case $k = 1$.

All posets P we consider are *graded* which means that every maximal chain C in P has the same size. The *rank* $r(x)$ of an element x in a graded poset P is one less than the maximum size of the chains which contain x as the top element. The *rank of* P, $r(P)$, is the maximum rank of any element in P. Let $P_i = \{x \in P : r(x) = i\}$. This partitions P into antichains, $P_0, \ldots, P_n$, where $n = r(P)$.

The sequence of Whitney numbers of a graded poset P is $(W_0, W_1, \ldots, W_n)$ where $W_i = |P_i|$ and $n = r(P)$. P is *rank-symmetric* if $W_i = W_{n-i}$ for all i, and *rank-unimodal* if for some j, $W_0 \leq W_1 \leq \cdots \leq W_j$ and $W_j \geq W_{j+1} \geq \cdots \geq W_n$.

The union of the k largest rank-sets P_i of P is a k-family so $d_k(P)$ is at least the sum of the k largest Whitney numbers W_i. P has the *strong Sperner property* if for all k, $d_k(P)$ actually equals this sum of the k largest Whitney numbers. A *Peck poset* is a rank-symmetric, rank-unimodal poset with the strong Sperner property (cf. survey [13]).

If a graded poset P of rank n has a partition $\boldsymbol{C}$ into chains such that each chain in $\boldsymbol{C}$ is saturated and symmetric about middle rank, $\frac{1}{2}n$, then P is called a *symmetric chain order*. This means that for each chain C in $\boldsymbol{C}$ there exists an i such that C consists of one element of each rank $i, i+1, \ldots, n-i$. Computing the bound $d(k, \boldsymbol{C})$, we find that $d(k, \boldsymbol{C}) = d_k(P)$ is the sum of the k middle Whitney numbers, so that P is a Peck poset. We also conclude that the partition $\boldsymbol{C}$ is *completely saturated*, which means it is k-saturated for all k.

Problems concerning subsets

Let $[n] = \{1, \ldots, n\}$. Let $B_n = (2^{[n]}, \subseteq)$ be the Boolean algebra of order n, which is the poset of all subsets of $[n]$, ordered by inclusion. Sperner's Theorem [22] states that B_n has width $\binom{n}{\lfloor \frac{1}{2}n \rfloor}$, the size of the middle rank(s). A stronger result, due to deBruijn et al. [4], is that B_n is a symmetric chain order (cf. [10, 11]).

1 (Z. Füredi [7]). Can B_n be partitioned into $\binom{n}{\lfloor \frac{1}{2}n \rfloor}$ chains of the same size (within one)? That is, letting a and b satisfy $2^n = a\binom{n}{\lfloor \frac{1}{2}n \rfloor} + b$, where $0 \leq b < \binom{n}{\lfloor \frac{1}{2}n \rfloor}$, we require b chains of size $a+1$ and $\binom{n}{\lfloor \frac{1}{2}n \rfloor} - b$ of size a. This is the minimum number of chains which can cover B_n, in light of Sperner's Theorem. This problem seems to be hard.

2 (B. Sands [21]). Can B_n be partitioned into chains of size 4 for sufficiently large n? More generally, for any k, can B_n be partitioned into chains of size 2^k for all sufficiently large n, given k?

In this question note that 2^k divides $|B_n| = 2^n$ for $n \geq k$, so a complete partition is possible. For $k = 1$ it is trivial to partition B_n for $n \geq 1$, e.g. take the chains $\{X, X \cup \{n\}\}$ for $X \subseteq [n-1]$. For $k = 2$, Griggs et al. [26] recently solved the original problem by showing that B_n can be partitioned into chains of size 4 if and only if $n \geq 9$. It is impossible for $n \leq 8$ since then the number of chains, 2^{n-2}, is less than $\binom{n}{\lfloor \frac{1}{2}n \rfloor}$. For $n = 9$ they have an ad hoc construction, which can be extended automatically by induction to all $n > 9$. However, the method does not extend to the general k problem. Of course it would be very nice if B_n could be partitioned into chains of size 2^k if and only if the number of chains $2^{n-k} \leq \binom{n}{\lfloor \frac{1}{2}n \rfloor}$.

We propose a stronger conjecture which involves chain sizes other than powers of 2. Given $c \geq 1$, can B_n be partitioned into chains of size c, except for at most $c - 1$ elements, which also belong to a single chain, for $n > n_0(c)$? This is trivial for $c = 1$ and is true for $c = 2, 4$ by the remarks above. For $c = 3$ it turns out to be true for all n.

3. Problems 1 and 2 seem promising particularly because Sperner theory is consitent with them. The strongest conjecture about chain sizes in partitions of B_n that is consistent with Sperner theory is this: Let $\boldsymbol{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be any partition of the integer 2^n into parts $\lambda_i \geq 0$. Let $\sigma = (\sigma_1 \geq \sigma_2 \geq \cdots)$ be the partition of 2^n corresponding to the sizes of the chains in the symmetric chain decomposition of B_n. Thus,

$$\sigma_1 = n + 1, \qquad \sigma_2 = \cdots = \sigma_{\binom{n}{1}} = n - 1, \qquad \sigma_{\binom{n}{1}+1} = \cdots = \sigma_{\binom{n}{2}} = n - 3,$$

and so on. A term $\sigma_i > 0$ if and only if $i \leq \binom{n}{\lfloor \frac{1}{2}n \rfloor}$.

The conjecture is that there is a partition of B_n with chain sizes $\boldsymbol{\lambda}$ if and only if $\boldsymbol{\sigma} \geq \boldsymbol{\lambda}$ in the majorization order, that is, $\sum \sigma_i = \sum \lambda_i$ and for all j

$$\sum_{i=1}^{j} \sigma_i \geq \sum_{i=1}^{j} \lambda_i.$$

The "only if" direction is a consequence of the symmetric chain decomposition being completely saturated. We have only checked the converse direction for very small cases, $n \leq 4$. The analogous conjecture for arbitrary posets is easily seen to be false, so this conjecture may be overly optimistic.

4. Determine partitions $\boldsymbol{\lambda}$ of 2^n such that B_n can be partitioned into *saturated* chains with sizes $\boldsymbol{\lambda}$. The symmetric chain decomposition described by $\boldsymbol{\sigma}$ above consists of saturated chains, but the conjecture for *saturated* chains analogous to problem 3 is false: There exists $\boldsymbol{\lambda} \leq \boldsymbol{\sigma}$ not realizable by a partition of B_n into saturated chains.

We recently obtained a partial answer to this problem [14]: Let $G(n, c, k) = \sum \{\binom{n}{i} : i \equiv k \pmod{c}\}$. If B_n is partitioned into saturated chains of size at most c (i.e. if $\lambda_1 \leq c$), then we show that

(i) the number of chains of size c is at most $G(n, c, \lfloor\frac{1}{2}(n+c)\rfloor)$ (i.e. $\lambda_i < c$ for $i > G(n, c, \lfloor\frac{1}{2}(n+c)\rfloor)$, and

(ii) the total number of chains is at least $G(n, c, \lfloor\frac{1}{2}n\rfloor)$ (i.e. $\lambda_i > 0$ for $i = G(n, c, \lfloor\frac{1}{2}n\rfloor)$.

Obtaining these bounds is easy, but it is interesting that both bounds are best-possible.

Problems for other posets

5 (R. Stanley [23, p. 182]). Let $L(m, n)$ denote the lattice of Ferrers diagrams fitting into an $m \times n$ rectangle, ordered by inclusion. Equivalently, $L(m, n)$ is the poset of all integer sequences $S = (0 \leq a_1 \leq \cdots \leq a_m \leq n)$ ordered by $S \leq S'$ if and only if $a_i \leq a_i'$ for $1 \leq i \leq m$. Is $L(m, n)$ a symmetric chain order? This was first shown for $m \leq 4$ and all n by Riess [20] and later rediscovered for $m = 3$ by Lindström [17] and for $m = 4$ by West [25]. Stanley proved in general that $L(m, n)$ is Peck (cf. [24, 15, 19] for later, more elementary proofs). Stanley [23] studied more generally a class of posets related to the Bruhat order of Weyl groups. He showed that these posets are Peck and asked whether they are symmetric chain orders.

6 (A. Björner [3, p. 189]). Is the weak ordering of the symmetric group, S_n, a symmetric chain order? In this ordering, a permutation $\boldsymbol{a} = (a_1, \ldots, a_n)$ covers $b = (b_1, \ldots, b_n)$ if $\boldsymbol{a}$ is obtained from $\boldsymbol{b}$ by transposing some adjacent pair b_i, b_{i+1} in $\boldsymbol{b}$ with $b_i < b_{i+1}$. For instance $(1, 4, 2, 3)$ covers $(1, 2, 4, 3)$ in S_4. The minimum element is thus $(1, 2, \ldots, n)$, the maximum element is $(n, n-1, n-2, \ldots, 1)$, and the rank of this ordering of S_n is $\frac{1}{2}(n(n-1))$. This ordering is rank-symmetric and rank-unimodal, but it is open whether it even has the Sperner property. The same questions are posed more generally for the weak ordering of a Coxeter group (W, S) with W finite [3].

7 (Folklore). Let $L_n(q)$ denote the lattice of subspaces of an n-dimensional vector space V over the finite field $\mathrm{GF}(q)$, ordered by inclusion. Then a subspace S of V has rank equal to its dimension. $L_n(q)$ is known to be a symmetric chain order [1, 12] by an existence proof which exploits the regularity of the lattice and combinatorial matching theory. The problem is to give an *explicit* symmetric chain decomposition analogous to the explicit ones for the Boolean lattice and the lattice of divisors of an integer. Such a result may require a nice method of describing the subspaces which would be helpful for other problems about $L_n(q)$, e.g. proving a subspace analogue of the Kruskal–Katona theorem.

8 (K. Engel [6]). An ordered partition $\boldsymbol{C}$ of a poset P into chains $C_1, \ldots, C_t$ is said to be *admissible* if $|C_1| \leqslant 3$ and if whenever two elements x and y belong to C_j and C_j has at least two elements above x and below y, there is some element z in $C_1 \cup \cdots \cup C_{j-1}$ such that $x < z < y$. The problem is to show that the product of three chains with equal length has an admissible ordered symmetric chain decomposition. This poset is known to be a symmetric chain order [4], so the problem is to find a decomposition which can be ordered appropriately. Engel found an admissible decomposition for the product of just two chains of equal length using a lovely zigzag construction. This result extends to every product of an *even* number of equal length chains by a product theorem for admissible partitions. To extend this to *any* product of at least two chains of equal length it suffices, by the product theorem, to solve the case of just three chains. This cannot follow simply from the product theorem since a single chain of size at least 4 is not admissible. Supporting evidence for the conjecture is the discovery by Mahnke [18] of admissible decompositions when the chain size is at most 5.

If the desired result is true, there would be a nice application to computing the minimum number of evaluations required to completely determine an unknown order-preserving map $f: P \rightarrow Q$ when P, Q are finite posets and P is a product of chains of equal length.

9 (Griggs [12]). A finite ranked poset P has the *LYM property* if for all $k > 0$ and all subsets $A \subseteq P_k$, the set ∂A of elements of P_{k-1} covered by some element of A satisfies

$$\frac{|\partial A|}{|P_{k-1}|} \geqslant \frac{|A|}{|P_k|}.$$

One class of LYM posets is the *regular* posets, which have the property that for all k, every element of P_k covers the same number α_k of elements of P_{k-1} and is covered by the same number β_k of elements of P_{k+1}. Anderson [1] and, independently, Griggs [12] proved that every rank-symmetric, rank-unimodal LYM poset is a symmetric chain order. The problem is to say something about LYM posets in general. Specifically, Griggs has conjectured since 1975 that LYM posets have completely saturated partitions. Since LYM posets P are known to be strong Sperner [10], this says equivalently that there is a partition $\boldsymbol{C}$ of P into chains such that whenever a chain C in $\boldsymbol{C}$ contains an element of any rank P_i it also contains an element of each rank P_j such that $|P_j| \geqslant |P_i|$. It might be easier to prove a considerably weaker result, e.g. that a rank-unimodal regular poset has such a completely saturated chain partition. However, no progress has been made on this problem since its formulation.

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A SOLUTION TO THE MISÈRE SHANNON SWITCHING GAME

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Let G be a graph and x_0, x_1 be two different vertices of G. Two players, Black and White, mark alternately non marked edges of G. White loses if and only if he marks all edges of a path connecting x_0 and x_1. This game is the misère version of the well-known Shannon Switching Game. We give its classification as a particular case of the classification of a more general game played on a matroid.

1. Introduction

Let M be a matroid and $e \in E(M)$. Two players, Black and White, mark alternately non marked elements of $E(M) - e$. In the ordinary Shannon Switching Game, White wins if he marks all elements of a circuit broken at e; otherwise Black wins. We recall that a *circuit broken at e* is a set of the form $C - e$, where C is a circuit of M containing e. We define the *Misère Switching Shannon Game on M with respect to e* as follows. The rules are the same as above except that White loses if he marks all elements of a circuit broken at e and wins otherwise.

The Shannon Switching Game was introduced by Shannon for graphs and generalized to matroids by Lehman in [4]. Lehman has given a complete classification of this game [4].

The Misère Shannon Switching Game is different from the misère game considered by Kano for graphs [5] and generalized to matroids in [3], where White loses if and only if he marks a circuit of M.

A matroid M is called a *block* if $E(M)$ is the union of two disjoint bases. A block of M is a subset $X \subseteq E$ such that the induced matroid $M(X)$ is a block.

Theorem 1A (Lehman [5]). *Let M be a block. Then White playing second can mark a base of M.*

Theorem 1B (Lehman [5]). *Let M be a matroid and $e \in E(M)$. The Shannon Switching Game with respect to e has the following classification.*

(i) *White wins playing second if and only if there is a block of M not containing e but spanning e in M.*

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(ii) *The first player wins if and only if there are blocks containing e in both M and M*.*

(iii) *Black wins playing second if and only if there is a block of M* not containing e but spanning e in M*.*

A survey of Lehman theory and short proofs of Theorem 1A and 1B are contained in [2].

2. A solution to the Misere Shannon Switching Game

We use the following lemma.

Lemma 2.1. *Let M be a matroid and X be a block of M. Two players Black and White play alternatively by marking elements of M. Then White playing first resp. second can force Black to mark a basis of $M(X)$.*

Proof. Since $M(X)$ is a block, then so is $(M(X))^*$. By Theorem 1A there is a strategy $\mathcal{S}$ for White playing second (resp. first) for marking a base of $(M(X))^*$.

White playing second uses the following strategy:

(1) If Black marks an element of $E(M)-X$, then White marks any element of $E(M)-X$ if there remains any. Otherwise White considers a fictitious move on X played by Black and answers according to $\mathcal{S}$.

(2) Black marks an element of X which is not a fictitious move. Then White answers according to $\mathcal{S}$.

(3) Blacks plays a fictitious move (i.e. an element chosen by White as a fictitious move of Black in the previous course of the game). Then White choses any non marked element if there remains any as a new fictitious move of Black and answers according to $\mathcal{S}$. If there is no such element the game is over.

Observe that up to the order of moves, on X White answered to all moves of Black according to $\mathcal{S}$. Hence White has marked a base of $(M(X))^*$. It follows that Black has marked a base of $M(X)$.

The case White playing first reduces to the case playing second by considering the game on M plus a loop played as first move by Black. □

The following definition is given in [1].

Let E be a set and $\mathscr{C} \in 2^E$. Two players, Black and White, mark alternately non marked elements of E. White wins if he marks all elements of a set in $\mathscr{C}$. Otherwise Black wins.

This game is a *positional game of type* 1 *with set of winning configurations* $\mathscr{C}$. We denote it by $(E, \mathscr{C})$.

Proposition 2.2. *Let $\mathscr{G}=(E, \mathscr{C})$ be a positional game of type* 1. *Suppose that White has a winning strategy playing second. Then White has a winning strategy playing first an arbitrary element $e \in E$.*

Proposition 2.2 is an easy refinement of the well-known fact that if White has a winning strategy playing second, then he has also a winning strategy playing first (see [1, 4]).

Theorem 2.3. *The Misère Shannon Switching Game on a matroid M with respect to an element $e \in E(M)$ has the following classification:*

(i) *If there is a block of M not containing e but spanning e in M then this game is winning for Black.*

(ii) *If there is a block of M and a block of M^* containing e, and $|E|$ is odd, then the first player wins.*

(iii) *If there is a block of M and a block of M^* containing e, and $|E|$ is even, then the second player wins.*

(iv) *If there is a block of M^* not containing e but spanning e in M^* then White wins.*

Proof. (i) Let X be a block of M spanning but not containing e. By Lemma 2.1, Black can force White to mark a base of $M(X)$. Then White loses.

(ii) Let X and X' be blocks of M and M^* containing e respectively. Note that $|E|-|X|$ and $|E|-|X'|$ are both odd.

(1) White playing first uses the following strategy: The first move is any element of $E\backslash X'$.

If Black marks an element of $E-X'$, then White marks any non marked element of $E-X'$ (always possible for parity reason).

If Black marks an element of X', then White plays according to the strategy given by Proposition 2.2 in the game on $(M^*(X'))^*$ with $\mathscr{C}$ the set of bases of $(M^*(X'))^*$ and e being his first move. Then when the game is over White has marked a base of $(M^*(X'))^*/e$, hence Black has marked a base of $M^*(X')$ not containing e but spanning e in M^*. It follows that there is no white circuit broken at e in M.

(2) Black playing first.

As in (1), Black forces White to mark a base of X. Then White marks a white circuit broken at e.

(iii) There is a block X of M and a block X' of M^* containing e and $|E|$ is even.

Note that $|E|-|X|$ and $|E|-|X'|$ are both even.

(1) White playing first.

Black uses the following strategy: If White marks an element of X, then Black plays according to the strategy given by Proposition 2.2 in the game on $(M(X))^*$ with $\mathscr{C}$ the set of bases of $(M(X))^*$ and e being his first move.

If White marks an element of $E-X$, then Black plays any element of $E-X$.

Using this strategy, Black forces White to mark a base of X. Hence White marks a circuit broken at e.

(2) Black playing first.

White uses the strategy in (1), where X is replaced by X' and $M(X)$ by $M^*(X')$.

(iv) Let X' be a block of M^* spanning but not containing e. by Lemma 2.1 White can force Black to mark a base of $M^*(X)$. It follows that there is no white circuit broken at e when the game is over. □

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A GRAPH-THEORETICAL CHARACTERIZATION OF THE ORDER COMPLEXES ON THE 2-SPHERE

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1. Background

Recently, some remarkable connections between commutative algebra and combinatorics have been discovered. Among the main topics in this area are the concepts of Cohen–Macaulay and Gorenstein complexes.

Recall some basic definitions. Let $V = \{x_1, x_2, \ldots, x_n\}$ be a finite set and Δ a *simplicial complex* on V, i.e. Δ is a collection of subsets of V satisfying (1) $\{v\} \in \Delta$ for all $v \in V$ and (2) $\sigma \in \Delta$, $\tau \subset \sigma$ imply $\tau \in \Delta$. The *dimension* of Δ is $(\max_{\sigma \in \Delta} \#(\sigma)) - 1$, where $\#(\sigma)$ is the cardinality of σ as a set. If $\#(\sigma) = i + 1$, then σ is called an *i-face*. Also, let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. We regard the elements x_i as indeterminates over k. Let I_Δ be the ideal of A generated by all square-free monomials not contained in Δ, namely

$$I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_r} \mid i_1 < i_2 < \cdots i_r, \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta).$$

Now define $k[\Delta]$ to be A/I_Δ, the quotient ring of A mod I_Δ. $k[\Delta]$ is called the *Stanley–Reisner ring* of the complex Δ.

We call Δ a *Cohen–Macaulay* (resp. *Gorenstein*) complex over k if the corresponding Stanley–Reisner ring $k[\Delta]$ is a Cohen–Macaulay (resp. Gorenstein) ring. The concept of Cohen–Macaulay or Gorenstein rings is very important in commutative algebra. Refer to Hochster [4] and Stanley [8] for the definition and elementary properties of Cohen–Macaulay and Gorenstein rings.

Now, the following theorem of Reisner is very important in the theory of Cohen–Macaulay complexes.

Theorem (Reisner [6]). *A complex Δ is Cohen–Macaulay over a field k if and only if for all $\sigma \in \Delta$ (including $\sigma = \emptyset$), $\tilde{H}_i(\mathrm{link}_\Delta(\sigma), k) = 0$ for $i < \dim(\mathrm{link}_\Delta(\sigma))$, where $\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta;\ \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Delta\}$.*

In particular, if the geometric realization $|\Delta|$ of Δ is a triangulation of a sphere, then Δ is Cohen–Macaulay over an arbitrary field k. Stanley [7] used this fact to prove the so called upper bound conjecture for spheres. Stanley's method is one of the most dramatic applications of commutative algebra to combinatorics.

Also, a finite poset P is called *Cohen–Macaulay* (resp. *Gorenstein*) over k if the order complex $\Delta(P)$ is Cohen–Macaulay (resp. Gorenstein) over k. Here, $\Delta(P)$ is the set of chains (totally ordered sets) of P, which turns out to be a simplicial complex on the set of elements of P. We define the *rank* of a poset P to be the dimension of $\Delta(P)$.

The theory of Cohen–Macaulay posets has been developed by many mathematicians, notably, Baclawski, Björner, Garsia and Stanley.

On the other hand, DeConciaí et al. (cf. [1]) introduced the concept of algebras with straightening laws. Let $R = \bigoplus_{n \geq 0} R_n$ be a commutative graded ring defined over a field k $(= R_0)$ and P a finite poset contained in R_1. Then a product $\alpha_1 \alpha_2 \cdots \alpha_n$ of elements of P in R is called a *monomial,* and a monomial $\alpha_1 \alpha_2 \cdots \alpha_n$ is called *standard* if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$.

Then R is called an *algebra with straightening laws* (abbreviated ASL) on P over k if the following conditions are satisfied:

(ASL-1) The set of standard monomials is a basis of R as a vector space over k.

(ASL-2) If α and β in P are incomparable and if

$$\alpha\beta = \sum_i r_i \gamma_i \delta_i \quad (0 \neq r_i \in k,\ \gamma_i \leq \delta_i)$$

is the unique expression for $\alpha\beta$ in R, as a linear combination of distinct standard monomials guaranteed by (ASL-1), then $\gamma_i \leq \alpha, \beta$ for every i.

A fundamental theorem on ASL's states that if P is Cohen–Macaulay (resp. Gorenstein) over k then every ASL on P over k is Cohen–Macaulay (resp. Gorenstein). From this fundamental theorem, we can obtain much information about any ASL on P from the combinatorics of the poset P.

Since we are interested in ASL's which are integral domains, we give the following definition:

Definition. A poset P is called *integral* over a field k if there exists an ASL domain on $P \cup \{\hat{0}\}$ over k, where $\hat{0}$ is a unique minimal element of $P \cup \{\hat{0}\}$. We use the convention that $\hat{0}$ is never an element of P.

For example, the cycle of length $2n$ (Fig. 1) is integral if and only if $n \leq 4$ (cf. [2]).

Next, we consider Gorenstein properties.

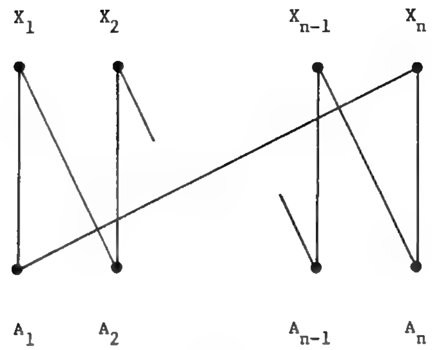


Fig. 1.

Definition. A Cohen–Macaulay poset P over a field k is called *weakly Gorenstein* over k if there exists a Gorenstein ASL on P over k.

We remark that if the geometric realization $|\Delta(P)|$ of the order complex $\Delta(P)$ of P is a triangulation of a sphere then P is Gorenstein, hence P is weakly Gorenstein.

Our final goal is to solve the following:

Problem. Characterize the weakly Gorenstein integral posets of rank $d-1$.

In the case of $d=1$, the answer is easy, namely, the weakly Gorenstein posets of rank zero are the poset of Fig. 2(a) or the poset of Fig. 2(b). Also, in the case of $d=2$, we can find all weakly Gorenstein integral posets.

Theorem (Hibi–Watanabe [2]). *Let k be an infinite field. Then the weakly Gorenstein integral posets over k are those shown in Fig.* 3.

However, if $d>2$ then the situation is very obscure. Now, to attack the case of $d=3$, to begin with, we must find all integral posets P such that $|\Delta(P)|$ is a triangulation of the 2-sphere. To do this, we consider the combinatorial problem of finding a simple way to classify all posets P for which $|\Delta(P)|$ is a triangulation of the 2-sphere.

Fig. 2.

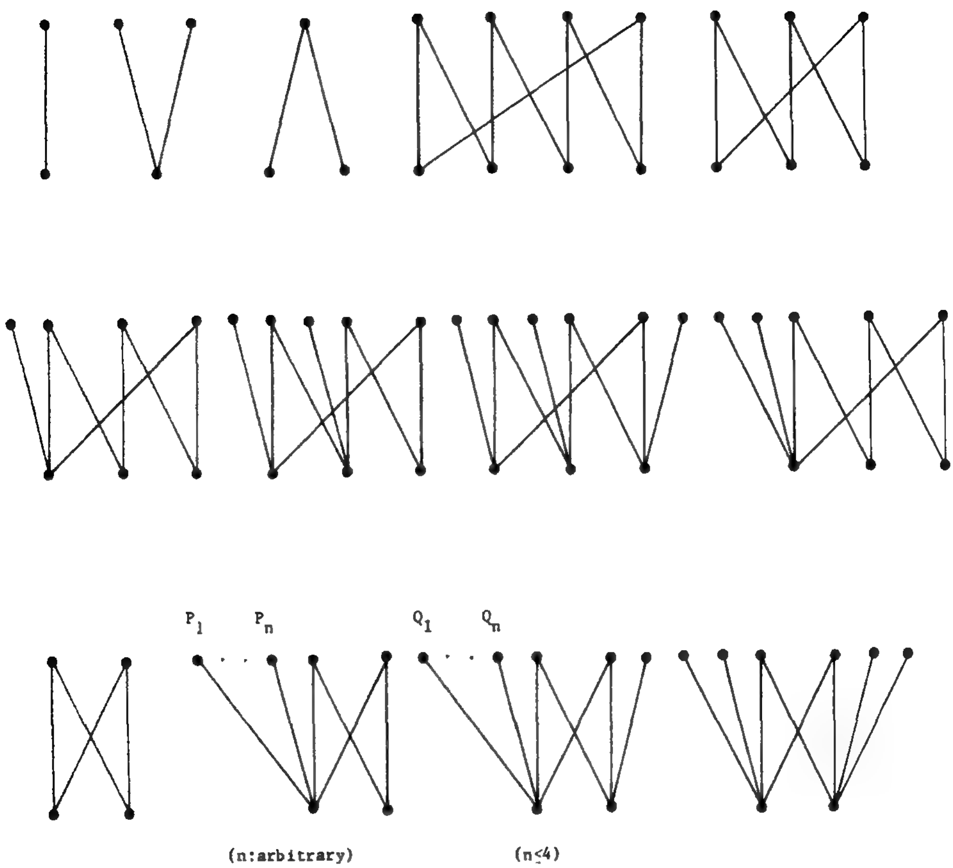


Fig. 3.

2. Main result

We now state our main result.

Theorem. *The following statements are equivalent for a poset P:*

(1) *The geometric realization of the order complex $\Delta(P)$ is a triangulation of the 2-sphere.*

(2) *P is the face poset of a 2-connected plane graph.*

To explain this theorem, we consider the following poset P (Fig. 4).

The 1-skeleton of $\Delta(P)$, written $\Delta(P)^{(1)}$, is regarded as a graph. And the geometric realization of $\Delta(P)$ is a triangulation of the 2-sphere (see Fig. 5). Therefore, for each β_j $(1 \leq j \leq 6)$, the number of 2-simplexes in $|\Delta(P)|$, which contain β_j, is four. $\Delta(P)^{(1)} - \{\beta_j \mid 1 \leq j \leq 6\}$ is regarded as a plane graph whose regions are quadrangles (see Fig. 6).

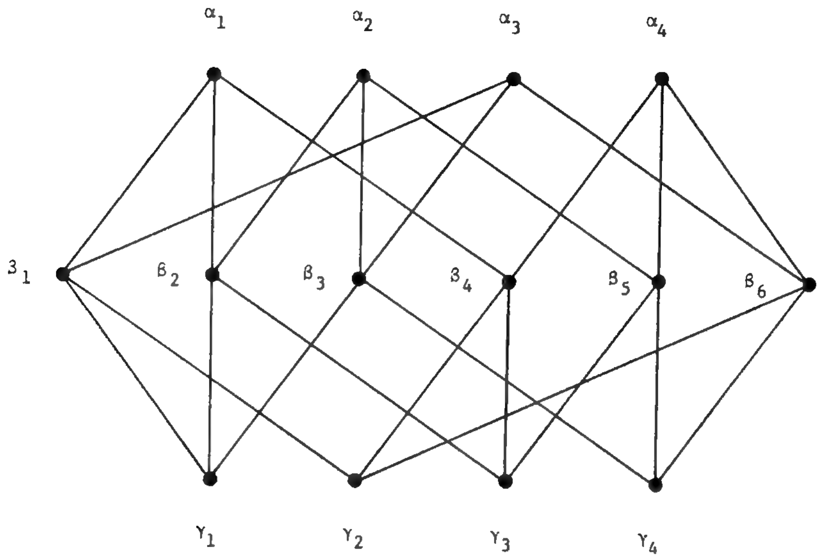
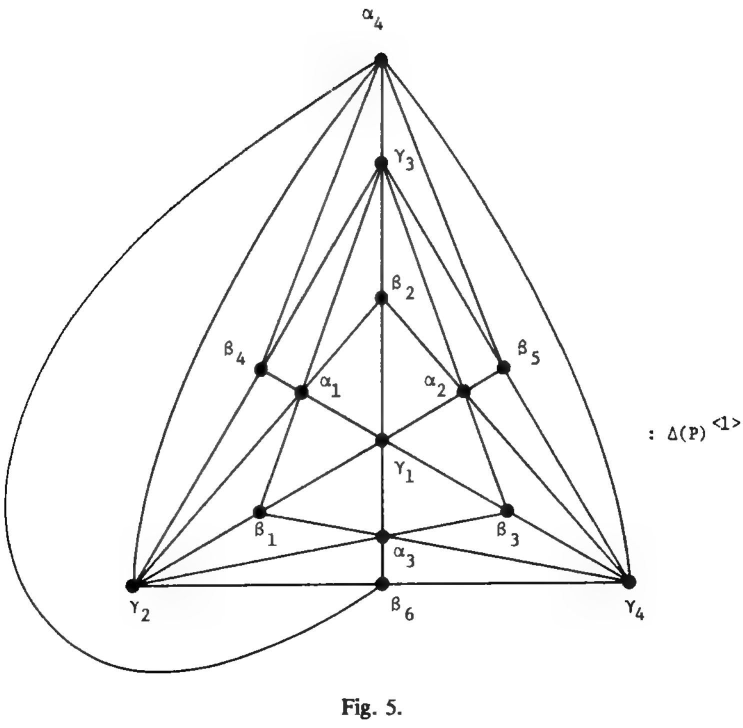

α_1 α_2 α_3 α_4
β_1 β_2 β_3 β_4 β_5 β_6
γ_1 γ_2 γ_3 γ_4

Fig. 4.


α_4
γ_3
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β_4
α_1
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: $\Delta(P)^{<1>}$
γ_1
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γ_4

Fig. 5.

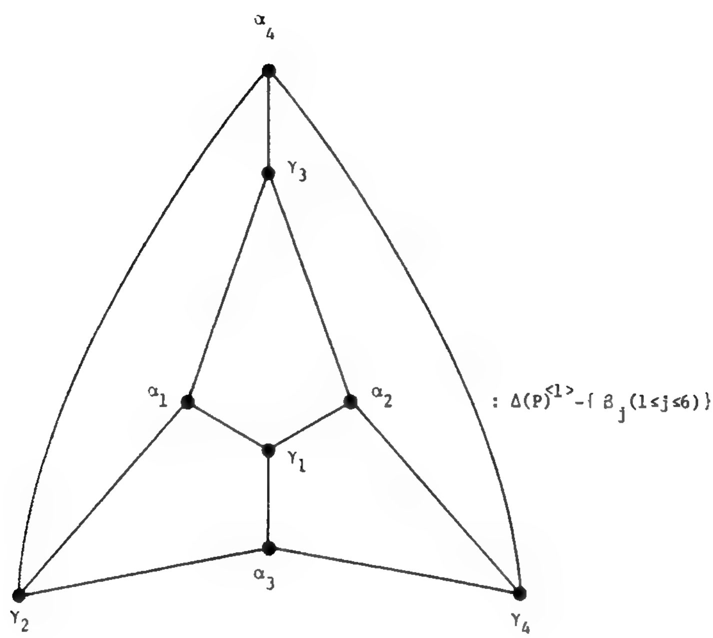

Fig. 6.

In the following, we shall construct a graph $G(P)$. The construction is in two stages:

(1) We choose $\{\gamma_m \mid 1 \leq m \leq 4\}$ as the vertex set of $G(P)$.

(2) For each quadrangle, we draw an edge joining the two vertices of $\{\gamma_m \mid 1 \leq m \leq 4\}$ in the quadrangle.

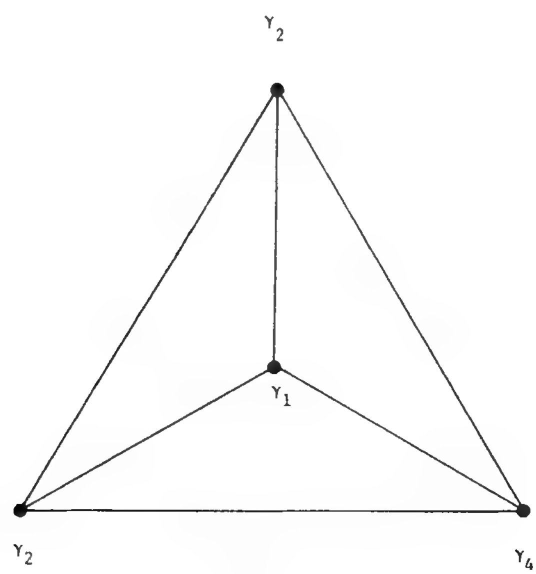

Fig. 7.

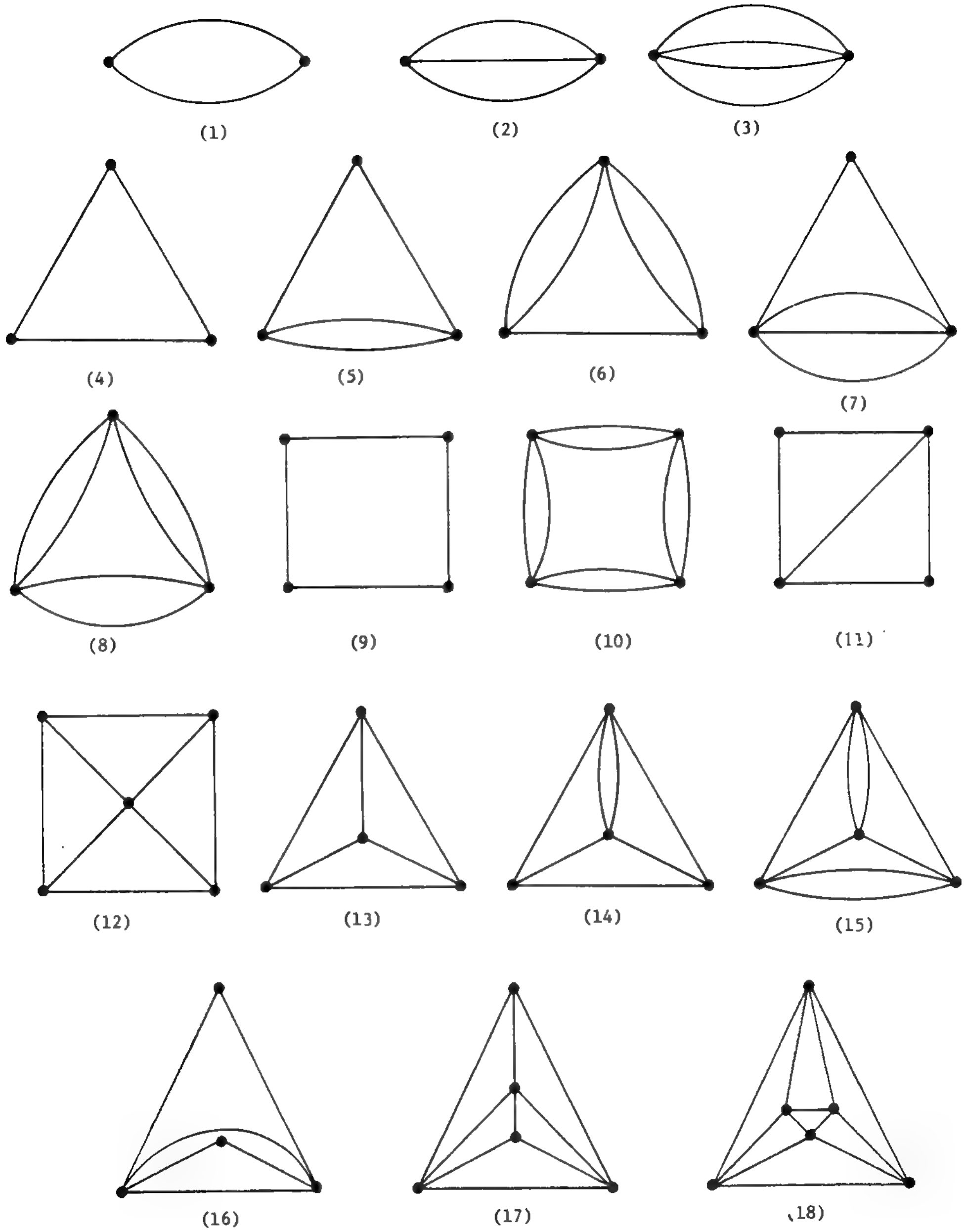

Fig. 8.

The graph $G(P)$ is a 2-connected plane graph and is regarded as a complex, where vertices, edges and regions of $G(P)$ are faces of the complex (see Fig. 7). The faces of $G(P)$ ordered by inclusion, form a poset. This poset is called the *face poset* of $G(P)$. Since each vertex, each edge and each region of $G(P)$ corresponds, respectively, to an element of $\{\gamma_m \mid 1 \leq m \leq 4\}$, an element of

$\{\beta_j \mid 1 \leq j \leq 6\}$ and an element of $\{\alpha_i \mid 1 \leq i \leq 4\}$, the poset P is the face poset of $G(P)$.

The above process gives the reader the basic idea behind the proof of our theorem. Consult [3] for further details.

Using our theorem, we can obtain a list of 2-connected plane graphs whose order complexes have at most 24 maximal faces, see [3]. Using this list K.-i. Watanbe proved the following.

Theorem (Theorem 2.2 in [9]). *Let k be a field and P a poset of rank two. Assume that the geometric realization $|\Delta(P)|$ of the order complex $\Delta(P)$ is a triangulation of the 2-sphere. Then P is integral over k if and only if $G(P)$ is isomorphic to one of the following* 18 *graphs* (*see Fig.* 8).

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HARD GRAPHS FOR THE MAXIMUM CLIQUE PROBLEM

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The maximum clique problem is one of the NP-complete problems. There are graphs for which a reduction technique exists that transforms the problem for these graphs into one for graphs with specific properties in polynomial time. The resulting graphs do not grow exponentially in order and number. Graphs that allow such a reduction technique are called soft. Hard graphs are those graphs for which none of the reduction techniques can be applied. A list of properties of hard graphs is determined.

1. Introduction

The maximum clique problem (MCP) is the problem to determine a largest clique in a graph G. It is one of the NP-complete problems, for which we refer to Garey and Johnson [1]. Graph theoretic terminology will be according to Harary [2]. The problem is equivalent to the independent set problem (ISP) in the complementary graph $\bar{G}$, which is the problem to determine a largest independent set.

We recall that a function $f(N)$, is defined on the set of natural numbers, is called $O(g(N))$ if $|f(N)| \leqslant m\,|g(N)|$ for $N \geqslant N_0$, where m and N_0 are non-negative constants. If a graph G has N points an algorithm has polynomial complexity, or is shortly called polynomial, if the number of steps in the calculation is $O(N^c)$, where c is a positive constant.

Definition 1. A graph G is called *soft* if either
 (i) there exists a polynomial algorithm to solve the MCP for G, or
 (ii) there exists a transformation of the graph, taking only polynomial time, into a polynomial number of graphs of polynomial order such that a specific property is removed.
Part (ii) of this definition is the new feature brought forward in this note.

All polynomials are understood to be in N, the number of points of the original graph. In most cases that we shall deal with these transformations involve reduction techniques in which the number of points, the order, or the number of lines, the size, of the graph that is to be investigated further is reduced.

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The philosophy behind this concept of soft graph can be made clear by the following example. Suppose a graph G is non-connected. Clearly we shall consider the components separately in our search for the maximum clique. By doing this we investigate graphs that have the property that they are connected. It takes only polynomial time to determine the components. There are only $O(N)$ components and their orders are $O(N)$ as well. These components may turn out to be hard in the sense that no polynomial algorithm is known for them. However, the original graph is soft in the sense that it has a property, non-connectedness, that certainly is not the deeper reason for the overall difficulty of the problem. We may determine the properties that are not the real cause for the overall difficulty. Their negations then determine more and more aspects of the structure of the graphs for which the problem is really difficult, the ***hard*** graphs for the maximum clique problem.

It is important to stress the fact that we are considering certain properties. As a result of some decomposition technique there may arise more graphs than one, each of which is to be investigated on maximum cliques. Such a decomposition not necessarily refers to a specific property. A typical example is the technique of Tarjan and Trojanowski [6].

Each point v determines a set of neighbours $N(v)$ and we may decompose the graph into N graphs $\langle v \cup N(v) \rangle$. All cliques are contained in these induced subgraphs. It is clear that on repetition of this procedure one finally finds a maximum clique. Obviously, as v is adjacent to all other points of $\langle v \cup N(v) \rangle$, v may be left out and one needs only determine a maximum clique in $\langle N(v) \rangle$. There is reduction of the order. However, each repetition introduces a factor $O(N)$ for the number of graphs that are to be investigated. A combinatorial explosion occurs and Tarjan and Trojanowski find an $O(2^{N/3})$ complexity. We shall call this procedure *direct decomposition*.

Reduction techniques like direct decomposition should be distinguished from a decomposition like splitting a graph into its components. That decomposition technique does not induce a combinatorial explosion for the simple reason that it cannot be repeated.

2. Soft graphs

The proofs of the following lemmas are quite simple and are available in a more elaborate version of this note. The proofs of Lemmas 2 and 10 are given as examples.

Lemma 1. *Non-connected graphs and graphs with non-connected complements are soft.*

Definition 2. Point v is said to *dominate* point w if $N(v) \supseteq N(w)$.

Lemma 2. *Graphs that contain a point that dominates some other point are soft.*

Proof. Finding a point that dominates some other point can be done by comparing $N(v)$ and $N(w)$ for all pairs of points v and w, i.e. in polynomial time. Let point v dominate point w, then any maximal clique that contains w has a counterpart clique in which w has been replaced by v. This clique may be maximal as well or may be part of an even large clique. For the search of a maximum clique point w may be deleted and the order of the graph is reduced. $O(N)$ repetitions of the procedure remove all dominated points. □

Lemma 3. *Graphs with cutpoints are soft.*

This lemma gives an example of a reduction technique in which the order of the graphs is reduced, but the total number of points in the resulting graphs, the blocks, has increased. As the number of resulting graphs is $O(N)$ the reduction technique does not induce a combinatorial explosion.

Definition 3. Cutgraphs of order k of a graph G are induced subgraphs on k points with point sets that are cut sets for G.

The obvious generalization of Lemma 3 is:

Lemma 4. *Graphs with cutgraphs of order smaller than or equal to a constant k are soft.*

A related result is that of Whitesides [7], that in our terminology reads:

Lemma 5. *Graphs with cutgraphs that are cliques are soft.*

If G is the line graph of a graph H, then cliques in G correspond to stars in H. Independent sets in G correspond to matchings in H. This was reason for the author to study decomposition methods based on turning graphs into line graphs [3]. As a result the following was found.

Lemma 6. *Graphs that, by addition of lines, can be turned into a line graph that is not a complete graph are soft.*

One of the forbidden induced subgraphs for a line graph is the graph $K_{1,3}$. This graph turns out to be non-relevant for the MCP. In [3] the author used detachments to turn graphs without induced subgraphs $K_4 - x$ into line graphs. However, if that line of thought is not followed there are even simpler proofs of the following lemma. One technique was given by Dahlhaus, another by Schrijver.

Lemma 7. *Graphs that do not contain $K_4 - x$ as induced subgraph are soft.*

The importance of graphs like $K_4 - x$ is that they have a signalling function with respect to the fact that a line belongs to only one clique. Whenever a line is uni-cliqual this is found out by the fact that the line does not occur as the diagonal of a $K_4 - x$. Based on a remark of H.J. Veldman we have:

Lemma 8. *Graphs with uni-cliqual lines are soft.*

The reader should note the special case in which a line does not belong to any triangle and therefore forms itself a maximal clique.

Triangles have a signalling function for uni-cliqual points. If all sets of three points that contain point v induce a K_3, point v is unicliqual.

Lemma 9. *Graphs with uni-cliqual points are soft.*

A, perhaps superfluous, note for the Lemmas 8 and 9 is that the number of cliques that is memorized is polynomial. No combinatorial explosion is induced in the space-complexity.

Another important induced subgraph on four ponts is $K_3 \cup K_1$, that we shall call a point-triangle (configuration). The complement of this graph is $K_{1,3}$.

Lemma 10. *Graphs without point-triangles as induced subgraphs are soft.*

Proof. If a graph G does not contain point-triangles then $\bar{G}$ does not contain a $K_{1,3}$ as induced subgraph. A deep result of Minty [4] and Sbihi [5] shows that the ISP can then be solved in polynomial time in $\bar{G}$, which solves the MCP for G. G is soft. □

This result can be used for a further reduction technique.

Lemma 11. *Graphs with points that have neighbourhood graphs that do not contain point-triangles are soft.*

However, a fixed number of point-triangles can be admitted, as we have:

Lemma 12. *Graphs with the property that each point v has in its neighbourhood graph $\langle N(v)\rangle$ a number of graphs $K_3 \cup K_1$, that is limited by a constant l, are soft.*

Discussion

The main point of further research is the extension of the list of properties that make a graph soft. One may hope for extension of the list of properties in such a

way that one can prove that no graphs can be found that have none of them. This would establish the fact that at least one of the reduction techniques is bound to succeed at each phase of the reduction process. In a situation like that the complexity of the composite algorithm consisting of successive reductions may be investigated. Meanwhile a set of hard graphs remains. The order of the smallest hard graph known, having none of the properties mentioned in the lemmas, is 12.

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DOUBLY REGULAR ASYMMETRIC DIGRAPHS

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1. Introduction

Let n, k and λ be positive integers such that there exists a symmetric (v, k, λ)-design $D = (P, B)$, where P and B denote the sets of points and blocks respectively. Further let us assume that there exists a bijection T from B to P satisfying the following two conditions:

(i) For every block α, $T(\alpha) \notin \alpha$.

(ii) If $T(\alpha) \in \beta$, then $T(\beta) \notin \alpha$.

Now we define a digraph $D = (V, A)$, where V and A denote the sets of vertices and arcs respectively, as follows: $V = P$, and (a, b) is an arc if and only if $b \in T^{-1}(a)$.

By properties (i) and (ii) of T, if (a, b) is an arc then $b \neq a$ and (b, a) is not an arc. Namely D is asymmetric.

For a vertex a put $N^+(a) = \{b \in V$ such that $(a, b) \in A\}$ and $N^-(a) = \{b \in V$ such that $(b, a) \in A\}$. Then $N^+(a) = \alpha$ where $T(\alpha) = a$, and $N^-(a) = \{T(\alpha)$ such that $a \in \alpha\}$. Since D is a symmetric design, we have that $|N^+(a)| = |N^-(a)| = k$, and $|N^+(a) \cap N^+(b)| = |N^-(a) \cap N^-(b)| = \lambda$, where $a, b \in V$ with $a \neq b$. Namely D is regular of valency k and doubly regular of double valency λ.

For brevity we call D a DRAD with parameters v, k and λ.

In this paper we show some basic properties of DRADs and consider the existence and non-existence of an above mentioned bijection T for given symmetric designs.

Notation. For a finite set S, $|S|$ denotes the number of elements of S.

2. Some basic properties of DRADs

Proposition 1. $v \geq 2k + 1$ *and hence* $k \geq 2\lambda + 1$.

Proof. If $v < 2k + 1$, then $N^+(a) \cap N^-(a) \neq \emptyset$, $a \in V$, This destroys the asymmetry of D. $\square$

Remark. The case where the equality holds in Proposition 1 corresponds to the case of Hadamard tournaments. For this see [1, 4 and 6].

Proposition 2. *D is strongly connected.*

Proof. Let a be any vertex and $C(a)$ the set of vertices to which there exist (directed) paths from a. By the definition of a DRAD it is easy to see that (i) there exists a vertex a such that $C(a) = V$ and (ii) $V = \{a \in V; C(a) = V\}$. □

Proposition 3. *Let a be any vertex, $N_2^+(a)$ the set of vertices whose distance from a equals two and $N_2^-(a)$ the set of vertices whose distance to a equals two. Then $|N_2^+(a)|$ and $|N_2^-(a)|$ are not less than $(n^2/2\lambda) + (\frac{1}{2}n) - (3\lambda/8)$, where $n = k - \lambda$.*

Proof. Let $N^+(a) = \alpha = \{b_1, b_2, \ldots, b_k\}$ and $b_i = T(\beta_i)$, $1 \leqslant i \leqslant k$. Then we have that $N_2^+(a) = \bigcup_{1 \leqslant i \leqslant k} \beta_i - \alpha$. Clearly it holds that $|\bigcup_{1 \leqslant i \leqslant x} \beta_i - \alpha| \geqslant xn - \binom{x}{2}\lambda$. Put $F(x) = xn - \binom{x}{2}\lambda$. Then $F(x)$ takes the largest value at $x = (n/\lambda) + (\frac{1}{2})$. So we have that $|N_2^+(a)| \geqslant (n/\lambda) - (\frac{1}{2}))n - (\frac{1}{2})((n/\lambda) - (\frac{1}{2}))((n/\lambda) - (\frac{3}{2}))\lambda = (n^2/2\lambda) + (\frac{n}{2}) - (3\lambda/8)$. The proof for $|N_2^-(a)|$ is similar. □

Proposition 4. *The girth g of a DRAD is equal to at most four.*

Proof. Take any vertex a. We show that $N_2^+(a) \cap (N^-(a) \cup N_2^-(a)) \neq \emptyset$. In fact, by Proposition 3 we have that $|\{a\}| + |N^+(a)| + |N_2^+(a)| + |N^-(a)| + |N_2^-(a)| - v \geqslant 1 + 2k + (n^2/\lambda) + n - (3\lambda/4) - v = (n/\lambda) + n + (\lambda/4) + 1 > 0$. If $N_2^+(a) \cap (N^-(a) \cup N_2^-(a)) = \emptyset$, then $N^+(a) \cap N_2^-(a) \neq \emptyset$. If $b \in N^+(a) \cap N_2^-(a)$, then there exists a vertex c such that (b, c) and (c, a) are arcs. This means that $c \in N_2^+(a) \cap N^-(a)$, which is a contradiction. □

Proposition 5. *The diameter d of a DRAD equals at most four.*

Proof. Suppose that $d \geqslant 5$. Then there exist six vertices a_i, $0 \leqslant i \leqslant 5$, such that $a_0 \to a_1 \to a_2 \to a_3 \to a_4 \to a_5$ is a shortest path from a_0 to a_5. This implies that $\{a_0\}$, $N^+(a_0)$, $N_2^+(a_0)$, $N_2^-(a_5)$, $N^-(a_5)$ and $\{a_5\}$ are pairwise disjoint. However, as in the proof of Proposition 4 we have that

$$|\{a_0\}| + |N^+(a_0)| + |N_2^+(a_0)| + |\{a_5\}| + |N^-(a_5)| + |N_2^-(a_5)| > v.$$

This is a contradiction. □

Question. Does there exist a DRAD with $g = 4$ or $d = 4$?

3. An existence theorem for DRADs

Proposition 6. *Let $D = (P, B)$ be a cyclic projective plane given as follows: $P = Z/(v)$, where Z denotes the ring of rational integers and $v = |P|$ and*

$B=\{\alpha+i, i\in P\}$, where α is any block (line) of D. Then we can define a bijection T from B to P satisfying (i) *and* (ii) *in Section* 1 *and*

(iii) $T(\alpha+i)=T(\alpha)+i$, $i\in P$.

We call such a bijection T cyclic.

Proof. We have that $v=|P|=n^2+n+1$ and $k=|\alpha|=n+1$. Hence we have that $|\alpha+\alpha|\leqslant n+1+\binom{n+1}{2}<n^2$ for $n\geqslant 4$. So if $n\geqslant 4$ let a be any point outside α such that $2a$ is outside $\alpha+\alpha$. Then define T as follows: $T(\alpha+i)=a+i$, $i\in P$. Apparently T satisfies (i) and (iii). If there exist two distinct elements i and j of P such that $T(\alpha+i)\in\alpha+j$ and $T(\alpha+j)\in\alpha+i$, then we have that $2T(\alpha)\in\alpha+\alpha$. This is a contradiction. If $n=2$ and $\alpha=\{0,1,3\}$, then define T as follows: $T(\alpha+i)=6+i$, $i\in P$. If $n=3$ and $\alpha=\{0,1,3,9\}$, then define T as follows: $T(\alpha+i)=4+i$, $i\in P$.

Proposition 7. *Let $D=(P,B)$ be a cyclic symmetric design given as in Proposition 6. (We do not assume that $\lambda=1$). Let a block α satisfy the following condition: $0\notin\alpha$ and $0\notin\alpha+\alpha$. Then we can define a cyclic bijection T as follows: $T(\alpha+i)=i$, $i\in P$.*

Proof. Obviously T satisfies (i) and (iii). If there exist two distinct elements i and j of P such that $T(\alpha+i)\in\alpha+j$ and $T(\alpha+j)\in\alpha+i$, then we have that $0=(i-j)+(j-i)\in\alpha+\alpha$, which is against the assumption.

In general, a symmetric design has more than one bijections T satisfying the assumptions in Section 1.

Example 1. Let us consider a cyclic projective plane with $n=3$ and $v=13$. Let $\alpha=\{0,1,3,9\}$ be a line. Then define T as follows: $T(\alpha)=2$, $T(\alpha+1)=5$, $T(\alpha+2)=8$, $T(\alpha+3)=7$, $T(\alpha+4)=9$, $T(\alpha+5)=12$, $T(\alpha+6)=10$, $T(\alpha+7)=1$, $T(\alpha+8)=6$, $T(\alpha+9)=11$, $T(\alpha+10)=3$, $T(\alpha+11)=0$ and $T(\alpha+12)=4$. It is easy to check that T satisfies (i) and (ii) in Section 1. Obviously T is not cyclic.

On the other hand, not every symmetric design has a bijection T satisfying the assumptions in Section 1.

Example 2. Let $D=(P,B)$ be a cyclic symmetric design with parameters $v=15$, $k=7$ and $\lambda=3$ such that $\{0,1,2,4,5,8,10\}$ is a block. Then there exists no bijection T for D satisfying (i) and (ii) in Section 1, though it is slightly laborious to check this fact by hand. We notice that D is a Hadamard design and that the corresponding Hadamard matric of order 16 is of the group type. For this see [2].

4. Remarks on adjacency and incidence matrices

Proposition 8. *Let $D = (P, B)$ be a symmetric design, where $P = \{a_1, a_2, \ldots, a_v\}$ and $B = \{\alpha_1, \alpha_2, \ldots, \alpha_v\}$, and let A be the incidence matrix of D. If A can be chosen so that $A + A^t + I$ is a $(0, 1)$ matrix, where t denotes the transposition and I is the identity matrix, then D admits a bijection T defined in Section 1 and vice versa.*

Proof. Suppose that D admits T. Then we may choose A so that the (i, j)-entry of A equals 1 if and only if $T(\alpha_i)$ belongs to α_j. Now the property (i) of T implies that the diagonal entries of A equal 0, and the property (ii) of T implies that if the (i, j)-entry of A equals 1 then the (i, j)-entry of A^t equals 0 for $i \neq j$. It is easy to prove the converse, too. □

Now it is well known that A is normal. See, for instance, [5]. However, we would like to remark that it is more than that, namely PAQ is normal for any permutation matrices P and Q. We call such a matrix π-normal.

Proposition 9. *Let A be a real square matrix of order v. Then A is π-normal if and only if A is normal and $AA^t = aI + bj$, where a and b are real numbers, and J denotes the all one matrix.*

Proof. If A is normal and $AA^t = aI + bJ$, then $PAQQ^tA^tP^t = PAA^tP^t = P(aI + bJ)P^t = aI + bJ = Q^t(aI + bJ)Q = Q^tA^tAQ = Q^tA^tP^tPAQ$.

Let $S = (s_{i,j})$ be a permutation matrix corresponding to a permutation σ, namely the (i, j)-entry of S equals 1 if and only if $j = \sigma(i)$, $1 \leq i \leq v$. If A is π-normal then $S^tA^tAS = ASS^tA^t = AA^t = A^tA$. Put $B = A^tA = (b_{i,j})$. Then the (i, l)-entry of S^tBS equals $\sum_{j,k} s_{j,i} b_{j,k} s_{k,l} = b_{\sigma^{-1}(i),\sigma^{-1}(l)}$. So we have that $b_i = b_{\sigma^{-1}(i),\sigma^{-1}(l)}$ for any permutation σ.

Now we return to the situation of Proposition 8 and assume that A has the property that $A + A^t + I$ is a $(0, 1)$-matrix. Then define T by $T(\alpha_l) = a_l$, $1 \leq 1 \leq v$. Then we get a DRAD D and A can be regarded as the adjacency matrix of D. Clearly k is a simple eigenvalue of A and the remaining $v - 1$ eigenvalues of A lies on the circle $|x| = (k - \lambda)^{\frac{1}{2}}$.

Finally we would like to mention the following question which is well known in the case of Hadamard designs.

Question. Suppose that a symmetric (v, k, λ) design D_1 exists, where $v \geq 2k + 1$. Then does there exist a symmetric design D_2 with the same parameters such that D_2 admits a bijection T satisfying the assumptions in Section 1?

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ORDERING OF THE ELEMENTS OF A MATROID SUCH THAT ITS CONSECUTIVE w ELEMENTS ARE INDEPENDENT

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Let M be a matroid on set E, $|E| = m$, with rank function r. For a positive integer w, M is said to be wth L-ind (C-ind) orderable if there exists an ordering O of E such that any consecutive (cyclically consecutive) w elements are independent. It is proved that M is wth L-ind orderable if and only if $\lfloor m/w \rfloor \cdot (w - r(E-S)) \leq |S| \leq \lceil m/w \rceil \cdot r(S)$ holds for any $S \subset E$. While, we conjecture that M is wth C-ind orderable if and only if $|S| \leq r(S) \cdot (m/w)$ holds for any $S \subset E$. This is verified for several classes.

1. Introduction

Let M be a matroid on set E with rank function r. We put $|E| = m$ and $r(E) = r$.

An ordering O of E is a bijection

$$O: E \to \{1, 2, \ldots, m\}.$$

For a positive integer w, O is said to be a wth L-ind (linearly independent) ordering if it satisfies the condition:

$$\text{(L)} A(i) = \{O^{-1}(k) \mid k = i, i+1, \ldots, i+w-1\}$$
$$\text{is independent for any } i, \quad 1 \leq i \leq m - w + 1.$$

While if ordering O satisfies the following strengthened condition (C), O is said to be the wth C-ind (cyclically independent) ordering.

$$\text{(C)} \ B(i) = \{O^{-1}(k) \mid k = i, i+1, \ldots, i+w-1 \pmod{m}\}$$
$$\text{is independent for any } i, \quad 1 \leq i \leq m.$$

If there exists a wth L-ind (C-ind) ordering, M is said to be wth L-ind (C-ind) orderable. Note that if M is ith L-ind (C-ind) orderable, then for any $j \leq i$ M is jth L-ind (C-ind) orderable. Furthermore, the dual of an rth C-ind orderable matroid is $(m-r)$th C-ind orderable.

The purpose of this paper is to characterize these matroids. For the wth L-ind orderable matroids, a complete characterization is obtained. The proof is a direct extension of the previous work [1], which considers the problem in case $w = r$.

However, for the wth C-ind orderable matroids, the problem looks very tough. We can only provide a necessary condition, called (w UNICOVER), though we conjecture that it is also sufficient. At present, we verify that the conjecture is true for the following cases.

(1) Graphs consisting of two disjoint spanning trees and any $w \leq r$.
(2) Matroids and $w = 1$ or 2.
(3) Simple graphs and $w = 3$ or 4.
(4) Complete graphs and any $w \leq r$.
(5) 2-trees and any $w \leq r$.

It must be noted that any matroid in (1), (4), and (5) satisfies (w UNICOVER) for any $w \leq r$.

2. wth L-ind orderable matroids

An independent set of cardinality w ($w \leq r$) is called for simplicity a w-set. Let $p = \lfloor m/w \rfloor$, $t = \lceil m/w \rceil$. The following two lemmas should be called the packing and covering theorems [2, 3] applied to the truncation of M to w, respectively.

Lemma 1. *There exist p disjoint w-sets in M if and only if the following condition is satisfied*:

$$(w\ PACK)\quad \text{For any } S \subset E,\ |S| \geq p \cdot (w - r(E-S)).$$

Lemma 2. *E is covered with t w-sets if and only if the following condition is satisfied*:

$$(w\ COVER)\quad \text{For any } S \subset E,\ |S| \leq t \cdot r(S).$$

The following lemma can also be proved by an analogous manner as for the above lemmas.

Lemma 3. *If M satisfies both (w PACK) and (w COVER), M is partitioned into p disjoint w-sets and one (possibly empty) independent set.*

Now we can give a characterization theorem which provides a polynomial time algorithm to recognize the wth L-ind orderable matroids.

Theorem 1. *The following four conditions for M are equivalent*:

(1) *Both (w PACK) and (w COVER) are satisfied, that is, for any* $S \subset E$,

$$p \cdot (w - r(E-S)) \leq |S| \leq t \cdot r(S)$$

(2) *E is covered with t w-sets and contains p disjoint w-sets.*
(3) *E is partitioned into p disjoint w-sets and one independent set.*
(4) *M is wth L-ind orderable.*

Proof. It suffices to prove (3)$\rightarrow$(4). Let $E = A \cup S_1 \cup \cdots \cup S_p$ be the partition as in (3) where S_i is a w-set and A, $0 \leq |A| < w$, is an independent set.

We first show a method of ordering $A \cup S_1$ such that its consecutive w elements are independent. Then the idea is applied to extend the ordering of $A \cup S_1 \cup S_2$, $A \cup S_1 \cup S_2 \cup S_3, \ldots$, and $A \cup S_1 \cup \cdots \cup S_p$.

An element which is assigned the kth order is denoted by e_k. First of all, we give A with arbitrary ordering. Let $|A| = q$ $(< w)$ and $A = \{e_1, \ldots, e_q\}$. $S_1 \cup \{e_q\}$ has at most one circuit and the circuit has non-null intersection with S_1, since A and S_1 are both independent. Choose any element from S_1 which is contained in the circuit if one exists, and any element otherwise. And let it be e_{q+w}. Then, $S_1^1 = S_1 \cup \{e_q\} - \{e_{q+w}\}$ is a w-set. $S_1^1 \cup \{e_{q-1}\}$ has at most one circuit which has non-null intersection with S_1-$\{e_{q+w}\}$ since S_1 and A-$\{e_{q+w}\}$ are both independent. Choose any element from S_1^1 which is contained in the circuit if one exists, and any element otherwise. Let the element be e_{q+w-1}. Then, $S_1^2 = S_1^1 \cup \{e_{q-1}\} - \{e_{q+w-1}\}$ is a w-set.

Continue this procedure up to get the ordering

$$\{e_{1+w}, \ldots, e_{q+w}\} \subset S_1.$$

The rest, S_1-$\{e_{1+w}, \ldots, e_{q+w}\}$, is ordered arbitrarily from $q+1$ through w. Thus we obtain an ordering of $A \cup S_1$. It is evident that every set of consecutive w element is a w-set.

The above ordering was based only on the facts that A and S_1 are both independent and that $|S_1| = w$. Furthermore, recall that in the procedure we are allowed to order A arbitrarily. Thus we see that the same principle applies to the ordering of $S_i \cup S_{i+1}$ when S_i is ordered and S_{i+1} is not $(i = 1, \ldots, p-1)$. Thus, the ordering is extended to $A \cup S_1 \cup \cdots \cup S_p$. $\square$

3. *w*th C-ind orderable matroids

If M is wth C-ind orderable, there exist m w-sets which cover E such that each element is covered exactly w times.

Consider a matroid $M^{(w)} = (E^{(w)}, r^{(w)})$ which is derived from M by replacing each element with w parallel elements. Note that $r^{(w)}(S^{(w)})$, $S^{(w)} \subset E^{(w)}$, is equal to $r(S)$ where $S \subset E$ consists of the elements whose corresponding elements in $M^{(w)}$ are in $S^{(w)}$. Then, if M is wth C-ind orderable, $M^{(w)}$ is covered with m w-sets and also contains m disjoint w-sets. Therefore, by Lemma 1 and 2,

$$\lfloor |E^{(w)}|/w \rfloor \cdot (r^{(w)}(E^{(w)}) - r^{(w)}(E^{(w)} - S^{(w)})) \leq |S^{(w)}| \leq \lceil |E^{(w)}|/w \rceil \cdot (r^{(w)}(S^{(w)})).$$

holds for any $S^{(w)} \subset E^{(w)}$. Since $|E^{(w)}| = wm$, this leads to

$$(m/w) \cdot (r - r(E - S)) \leq |S| \leq (m/w) \cdot r(S) \quad \text{for any } S \subset E.$$

In contrast to the linear case, the above two inequalities are equivalent [4]. Thus we get a necessary condition of the wth C-ind orderable matroids which is able to

be checked in polynomial time [2]. We believe that this condition is also sufficient. Indeed, we have various affirmative examples for the conjecture, which we describe in primal form in the following.

Conjecture. A matroid M is wth C-ind orderable if and only if

$$(w\ \text{UNICOVER}) \quad |S| \leqslant (m/w) \cdot r(S) \quad \text{for any } S \subset E$$

is satisfied.

Note that M satisfies (r UNICOVER) if and only if its dual satisfies (r UNICOVER).

In the following, we list several matroids satisfying the condition for which we have been able to give wth C-ind orderings.

4. Examples of *w*th C-ind orderable matroids

For a graph $G = (V, E)$, $G - S$, $S \subset V$, is the graph obtained from G by deleting S. $E(G)$ also denotes the set of edges of G. An edge connecting vertices v and u is denoted by (v, u).

A graph consisting of two disjoint spanning trees is called a CTS-graph (complementary tree structure graph) and one of the trees is called a peripheral tree. It is easy to see that any CTS-graph satisfies (w UNICOVER) for any $w \leqslant r$.

Theorem 2 [1]. *A CTS-graph is wth C-ind orderable for any $w \leqslant r$.*

Proof. It suffices to prove the theorem for $w = r$. We shall apply induction on r. If $r = 1$ then the theorem is obviously true. (See Theorem 3.) Suppose that $r \geqslant 2$. Note that G contains a vertex of degree 2 or 3. If there exists a vertex of degree 2, say v, G-$\{v\}$ is again a CTS-graph. By induction hypothesis, there is an $(r-1)$th C-ind ordering O' in G-$\{v\}$. Define an ordering O of edges of G as follows.

$$O(e) = \begin{cases} O'(e) & (e \in E(G - \{v\}) \text{ and } O'(e) \leqslant r-1) \\ O'(e)+1 & (e \in E(G - \{v\}) \text{ and } O'(e) \geqslant r) \\ r & (e \text{ is one edge incident to } v) \\ 2r & (e \text{ is the other edge incident to } v). \end{cases}$$

It is easy to see that any cyclically consecutive r edges under O are independent.

Suppose that there is a vertex of degree 3, say v. Let x, y, and z be the vertices adjacent to v. Two of them may be identical. Assuming that (x, v) and (y, v) are contained in a peripheral tree, let G' be the graph obtained from G by removing vertex v and adding an edge (x, y). G' is also a CTS-graph. Hence there is an

$(r-1)$th C-ind ordering O'. Without loss of generality, we can assume $O'((x, y)) = 1$. Let $P = \{O'^{-1}(2), O'^{-1}(3), \ldots, O'^{-1}(r-1)\}$. Since $P \cup \{(x, y)\}$ is a spanning tree of G', exactly one of x and y, say x, is connected to z in P. Now define an ordering O of edges of G as follows.

$$O(e) = \begin{cases} O'(e) & (e \in E(G') \text{ and } 2 \leqslant O'(e) \leqslant r-1) \\ O'(e)+1 & (e \in E(G') \text{ and } O'(e) \geqslant r) \\ 1 & (e = (v, a)) \\ r & (e = (v, c)) \\ 2r & (e = (v, b)). \end{cases}$$

It is easy to see that O is an rth C-ind ordering. □

Since it is trivial that matroid M satisfies (1 UNICOVER) if and only if M contains no loop, the following fact is evident.

Theorem 3. *A matroid M is* 1*st C-ind orderable if and only if M satisfies* (1 *UNICOVER*).

Since it is easy to see that matroid M satisfies (2 UNICOVER) if and only if $r \geqslant 2$ and any $S \subset E$ such that $r(S) = 1$ contains at most $\frac{1}{2}m$ elements, it is not difficult to prove the following theorem.

Theorem 4. *A matroid M is* 2*nd C-ind orderable if and only if M satisfies* (2 *UNICOVER*).

Lemma 4. *A simple graph G satisfies* (3-*UNICOVER*) *if and only if $r \geqslant 3$ and G is not the graph with $m = 4$ containing a triangle.*

Proof. The necessity is obvious. Suppose that G does not satisfy (3-UNICOVER). If $r \geqslant 3$, there exists S such that

$$|S| > (m/3) \cdot r(S).$$

Since $m \geqslant |S|, 2 \geqslant r(S) \geqslant 1$. If $r(S)$ is 1 or 2, then $|S|$ is at most 1 or 3, respectively. By the inequality, $m \leqslant 2$ or $m \leqslant 4$, respectively. From $r \geqslant 3$ and $m \geqslant r + |S| - r(S)$, only the case when the above inequality holds is $m = 4$, $r(S) = 2$, and $|S| = 3$. □

Theorem 5. *A simple graph G is* 3*rd C-ind orderable if and only if G satisfies* (3 *UNICOVER*).

Proof. The theorem is obviously true when $r = 3$. Furthermore, it is not difficult to check that every simple graph is 3rd C-ind orderable if $r = 4$. With using this

fact as basis, we prove the theorem by induction on r. Suppose that $r(G) \geqslant 5$, $|V| = n \geqslant 6$. Let δ be the minimum degree of G. Then, $\delta \leqslant \frac{1}{3}m$ since $\delta n \leqslant 2m$. Let v be a vertex of degree at most $\frac{1}{3}m$. By the induction hypothesis, there exists a 3rd C-ind ordering in G-$\{v\}$. We can insert the edges which are incident to v into this ordering so that no pair of these edges are within distance 3 to get a 3rd C-ind ordering of G. $\square$

The following lemma seems not so trivial but the proof is omitted here for the space.

Lemma 5. *A simple graph G satisfies* (4 *UNICOVER*) *if and only if $r \geqslant 4$ and G does not contain K_3 if $m = 5$, $K_{1,1,2}$ if $m = 6$, or K_4 if $m = 7$ as an induced subgraph.*

Theorem 6. *A simple graph $G = (V, E)$ is 4th C-ind orderable if and only if G satisfies* (4 *UNICOVER*).

Proof. It is so cumbersome to describe our proof in detail that we will sketch it. First, we prove that any connected simple graph with 7 vertices is 4th C-ind orderable, almost through the exhaustive way. Then, we can prove the fact that any connected simple graph which has more than 7 vertices is 4th C-ind orderable, completing the proof. This key fact is proved by induction on n as follows. Suppose that G has n (>7) vertices. From $\delta n \leqslant 2m$ and $n \geqslant 8$, it is concluded that G contain a vertex of degree at most $\frac{1}{4}m$, say v. By induction hypothesis, the edges of G-$\{v\}$ can be so ordered that any cyclically consecutive 4 edges are independent. We can insert the edges which are incident to v into this ordering to get an ordering of E so that no pair of these edges are within distance 4. $\square$

It is easy to see that a complete graph satisfies (r UNICOVER). Illustrative examples in Fig. 1 and 2 showing a way of rth C-ind ordering will be the proof of the following theorem.

Theorem 7. *A complete graph is wth C-ind orderable for any $w \leqslant r$.*

For a positive integer k, a k-tree is the graph defined recursively as follows: (1) K_k is a k-tree, (2) a graph obtained from a k-tree T_k by adding a vertex v and edges that connect v and the vertices forming a K_k in T_k is a k-tree.

Lemma 6. *A k-tree satisfies* (*r-UNICOVER*).

The proof is omitted here for reasons of space. But for the case of 2-trees, the following theorem implies the lemma.

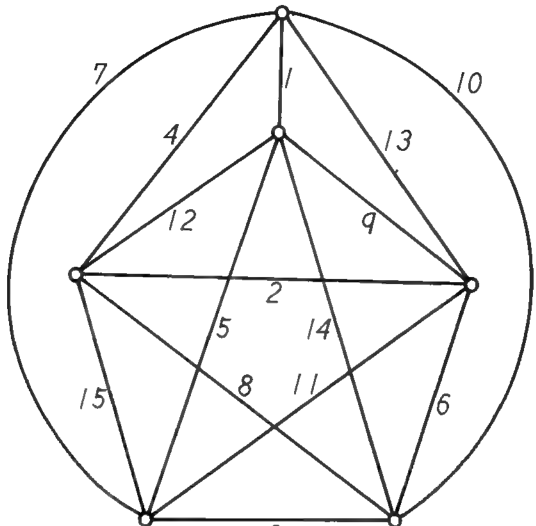

Fig. 1. An rth C-ind ordering of K_6.

Theorem 8. *A 2-tree is wth C-ind orderable for any* $w \leqslant r$.

Proof. We shall apply induction on r. If $r = 1$, that is K_2, then our theorem is true. Suppose that $r \geqslant 2$. By the definition of 2-tree, we can find a vertex v which was added at the last stage of construction. Let u and w be the vertices which are adjacent to v. G-$\{v\}$ is a 2-tree of rank $r-1$, and so is $(r-1)$th C-ind orderable.

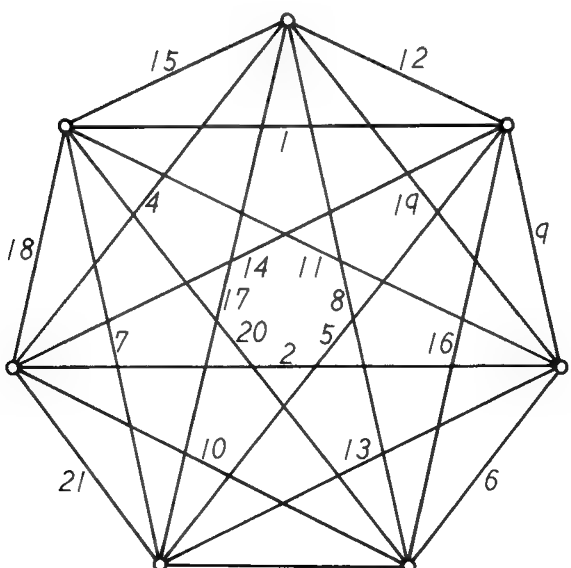

Fig. 2. An rth C-ind ordering of K_7.

Let O' be such an ordering and $O'((u, w)) = 1$. Define an ordering of edges of G as follows:

$$O(e) = \begin{cases} 1 & (e = (u, w)) \\ 2 & (e = (u, v)) \\ O'(e) + 1 & (e \in E(G - \{v\}) \text{ and } 2 \leq O'(e) \leq r - 1) \\ r + 1 & (e = (w, v)) \\ O'(e) + 2 & (e \in E(G - \{v\}) \text{ and } r \leq O'(e) \leq 2r - 3). \end{cases}$$

Every set of cyclically consecutive r edges under O that contains just one of (u, v) and (w, v) is independent. The only exception is set $S = \{(u, v), O^{-1}(3), \ldots, O^{-1}(r), (w, v)\}$. This is also independent because $S \cup \{(u, w)\} - \{(u, v), (w, v)\}$ is an $(r-1)$-set of G-$\{v\}$. $\square$

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AN EXISTENTIAL PROBLEM OF A WEIGHT-CONTROLLED SUBSET AND ITS APPLICATION TO SCHOOL TIMETABLE CONSTRUCTION

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1. Introduction

A school timetable problem is to find an efficient algorithm for constructing a school timetable which attains various given requirements consistently. Although many theoretical models (Cole [4], Csima [5], Gotlieb [9], Ikeda [13], Neufeld [24], Salazar [25], Welsh [26]) and construction algorithms (Almond [1], Csima [6], Dempster [7], Ikeda [10], Kirchgassner [20], Lions [21], Yule [29]) of school timetables have been proposed, the problem is still an unsolved one in both practical and theoretical point of view.

In theoretical sense, those models are not sufficient to represent various complicated requirements for school timetables. Even if almost all requirements could be represented in a model, the models were so complicated that it was very difficult to find an efficient and simple algorithm. There are some theoretical solutions of timetable problems under very limited requirements. But it is nonsense in a practical point of view, because schools could not accept such timetables in the actual education.

On the other hand, there are some computer programs that help to construct a school timetable. Alsmost all of these programs simulate traditional manual ways and cannot investigate the constructability of a timetable in advance.

In this paper, we propose a new combinatorial problem including a school timetable problem and give some theoretical solutions. The problem seems very simply but general because it also includes a network flow problem, a graph coloring problem and some other combinatorial problems as a special case of it.

2. School timetable problem

A school timetable problem was formalized in various manners. But almost all formalizations are too simple to accept a timetable constructed with them for actual schools. From the practical point of view, an actual school timetable problem should be formalized with the following complexity.

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Definition 1 (Specification of timetable problem). A *school timetable problem* is an 11-tuple $\langle T, C, W, p, E, L, g, h, r, I, A_0\rangle$ which satisfies the following conditions:

(1) T is a finite set which is called the set of all teachers,
(2) C is a finite set which is called the set of all classes,
(3) W is usually a set {MON, TUE, WED, THU, FRI} and sometimes includes "SAT". It is called the set of all names of days in a week.
(4) p is a function $W \rightarrow \{1, 2, \ldots\}$, where $p(w)$ means number of periods in day w,
(5) E is a finite set which is called the set of all special rooms,
(6) L is a subset of $(2^T - \{\emptyset\}) \times (2^C - \{\emptyset\}) \times (2^E) \times \{1, 2, \ldots\}$ which is called the set of all different lessons,
(7) g is a function from $T \times W$ to $\{1, 2, \ldots\}$, where $g(t, w) = m$ means that teacher t can give upto m lessons on day w,
(8) h is a function from E to $\{1, 2, \ldots\}$, which defines capacity of each special room,
(9) r is a function from L to $\{1, 2, \ldots\}$, where $r(l) = m$ means that a lesson l should be given m times in a week,
(10) I is a subset of $T \times C \times E \times W \times \{1, 2, \ldots\}$, which gives the set of all combinations to be inhibited,
(11) A_0 is a subset $L \times W \times \{1, 2, \ldots\}$, where $(l, w, p) \in A_0$ means that lesson l should be given from period p on day w.

In a high school (in Japan), each class has each class room. Set E is the set of all special rooms, e.g. music rooms, labs, gymnasiums other than ordinary class rooms. An ordinary lesson is specified by a teacher, a class and an ordinary class room or sometimes a special room.

A lesson which has two or more teachers and/or classes is called *a set of parallel lessons*, or simply a *para-lesson*. An example of a para-lesson l is a lesson of art that consists of three lessons, that is, music l_1 by teacher t_1 using music room e_1, painting l_2 by teacher t_2 using painting room e_2, and writing l_3 by teacher t_3 using writing room e_3. Each student of two classes c_1 and c_2 takes one of these three lessons. In order that lesson l makes no effect on timetable construction based on assignment of each class-teacher combination to a period, three lessons (t_1, c_1, e_1), (t_2, c_2, e_2) and (t_3, c_*, e_3) must be assinged at the same period where c_* is a dummy class introduced imaginary. In such a case, a lesson includes two or more teachers, classes and special rooms as a result.

An ordinary lesson occupies one period. There may be a special lesson which occupies two or more consecutive periods. Such a lesson is called a *long-lesson*. Number of periods occupied by a long-lesson is specified by the last component of l. If the last component is 1, it means that the lesson occupies one period.

There may exist requirements that cannot be expressed within the above

framework. For example, a period before a lession of physics must be opened for a certain teacher, because the period is sometimes used for preparation of experiment. Another example is a lesson of cooking which should be assigned to 4th or 5th period (immediately before or after lunch time) because students would have dishes made by themselves. These requirements are, however, relatively minor and there are many other cases when these can be avoided by some pre-assignments in A_0 or pre-inhibitions in I. But in a practical school timetable construction, it is at least necessary to formalize as Definition 1.

In terms of timetable problem defined above, a solution, called a *timetable*, can be defined as follows:

Definition 2 (Solution of timetable problem). A *solution* of a timetable problem $\langle T, C, W, p, E, L, g, h, r, I, A_0\rangle$ is a subset of A of $L \times W \times \{1, 2, \ldots\}$, satisfying the following conditions:

(1) If $(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i)$ is in A then $i + n - 1 \leq p(w)$,

(2) If two elements of A in the same day w, say $(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i)$ and $(\boldsymbol{t}', \boldsymbol{c}', \boldsymbol{e}', n', w, i')$ which include common teachers, i.e. $\boldsymbol{t} \cap \boldsymbol{t}' \neq \emptyset$, then two period sets $\{i, i+1, \ldots, i+n-1\}$ and $\{i', i'+1, \ldots, i'+n'-1\}$ are disjoint,

(3) If two elements of A in the same day w, say $(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i)$ and $(\boldsymbol{t}', \boldsymbol{c}', \boldsymbol{e}', n', w, i')$ which include common classes, i.e. $\boldsymbol{c} \cap \boldsymbol{c}' \neq \emptyset$, then two period sets $\{i, i+1, \ldots, i+n-1\}$ and $\{i', i'+1, \ldots, i'+n'-1\}$ are disjoint,

(4) For each teacher t and each day w, number of periods occupied by lessons that teacher t gives in day w is less than or equal to $g(t, w)$, that is,

$$\sum_{(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in A \text{ and } t \in \boldsymbol{t}} n \leq g(t, w),$$

(5) For each special room e and each period j of day w, number of lessons which use room e at period j is less than or equal to $h(e)$, that is,

$$|\{\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in A : e \in \boldsymbol{e} \text{ and } j \in \{i, i+1, \ldots, i+n-1\}| \leq h(e),$$

(6) For each lesson $l = (\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n)$ in L, number of elements in A which support it is equal to $r(l)$, that is,

$$|\{(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in A\}| = r(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n),$$

(7) For each element (t, c, e, w, i) in I, there is no element $(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i)$ of A such that $t \in \boldsymbol{t}$, $c \in \boldsymbol{c}$ and $e \in \boldsymbol{e}$,

(8) Set A_0 is a subset of A.

If a timetable problem specified in Definition 1 would be solved completely within a limited time, the timetable could be accepted in actual schools. The formalization is, however, too complex to find simple and efficient algorithms.

3. An existential problem of an n-dimensional weighted marginal-bounded 0–1 matrix and Gotlieb's model of a school timetable problem

In order to formalize a timetable problem, Gotlieb [9] proposed a combinatorial problem as follows:

Definition 3 (Specification of nWMP). An *existential problem of an n-dimensional weighted marginal-bounded* 0–1 *matrix* is a 3-tuple $\langle M, a, b\rangle$ satisfying the following conditions:

(1) M is a function from N_0 to N,

(2) a is an n-dimensional vector $\langle a_1, a_2, \ldots, a_n\rangle$ of function a_i from N_i to N for $i = 1, 2, \ldots, n$,

(3) b is an n-dimensional vector $\langle b_1, b_2, \ldots, b_n\rangle$ of function b_i from N_i to N for $i = 1, 2, \ldots, n$,

where $N = \{0, 1, \ldots\}$,

$$N_0 = \{1, 2, \ldots, m_1\} \times \{1, 2, \ldots, m_2\} \times \cdots \times \{1, 2, \ldots, m_n\},$$
$$N_i = \{1, 2, \ldots, m_1\} \times \{1, 2, \ldots, m_2\} \times \cdots \times \{1, 2, \ldots, m_{i-1}\} \times \{1, 2, \ldots, m_{i+1}\} \times \cdots \times \{1, 2, \ldots, m_n\}.$$

The existential problem of an n-dimensional weighted marginal-bounded 0–1 matrix is also referred to *nWMP* in short.

Definition 4 (A solution of nWMP). A *solution A* of an nWMP $\langle M, a, b\rangle$ is a function from M to $\{0, 1\}$ which satisfies the following conditions:

$$a_i(x) \leq \sum_{y \in N\{(i)=x\}} A(y) \cdot M(y) \leq b_i(x)$$

for any x in N_i and any i in $\{1, 2, \ldots, n\}$,

where $N\{(i) = x\} = \{(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in N_0\colon (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in N_i\}$.

Function M is called a *weight function* and vectors a and b are called a *lower bound* and an *upper bound* respectively. A special case when $M(x) = 1$ or 0 for any x in N_0 is sometimes referred to as *nMP*.

We shall give an example of nWMP in a case that $n = 2$, $m_1 = 8$ and $m_2 = 5$. A weight function M and bounds a and b are given in Fig. 1, where (i, j)th element is m, means $M(i, j) = m$ and ith element of the column (or row) vector named "a" and "b" indicates $a_1(i)$ (or $(a_2(i))$ and $b_1(i)$ (or $b_2(i)$) respectively. A solution A of the problem is also given in Fig. 1.

In a case that $n = 2$, nWMP is solvable by embedding it into a network flow problem. But if $n > 2$, nWMP and even nMP and now unsolved problems.

M	1	2	3	4	5	a	b	A	1	2	3	4	5
1	0	2	0	0	0	0	2	1	0	1	0	0	0
2	1	0	2	0	1	1	2	2	0	0	0	0	1
3	1	1	0	0	0	1	1	3	1	0	0	0	0
4	3	0	0	1	0	0	1	4	0	0	0	1	0
5	1	5	0	0	1	1	5	5	0	1	0	0	0
6	1	0	0	1	2	1	3	6	0	0	0	1	0
7	1	0	0	0	0	0	1	7	1	0	0	0	0
8	0	1	3	1	4	1	3	8	0	0	0	1	0
a	2	7	0	2	1								
b	3	8	2	3	4								

Fig. 1. An example of nWMP and its solution.

Gotlieb [9] pointed out the relationship between 3MP and a timetable problem as follows:

Let $\langle T, C, W, p, E, L, g, h, r, I, A_0\rangle$ be a timetable problem that satisfies the following conditions:

(1) Each lesson is an ordinary one, i.e.,

$$\{L \in (t, c, e, n) : |t| = 1,\ |c| = 1,\ |e| = 0,\ n = 1\},$$

(2) There is no constraint on number of lessons in each day w for each teacher t, i.e.

$$g(t, w) = p(w),$$

(3) There is no constraint on number of lessons using each room e at each period, i.e.

$$h(e) = |C|,$$

(4) There is no pre-assignment, i.e.

$$A_0 = \emptyset.$$

Such a timetable problem is assumed to be *simple* in this paper.

Theorem 1. *A simple timetable problem can be embedded into* 3*MP*.

Proof. Assume that sets T, C and $\{(w, i) : w \in W,\ i \in \{1, 2, \ldots, p(w)\}\}$ (say, P) are integer sets, $\{1, 2, \ldots, m_T\}$, $\{1, 2, \ldots, m_C\}$ and $\{1, 2, \ldots, m_P\}$ respectively. A function M from $T \times C \times P$ to the set of all integers is defined as,

$$M(t, c, i) = \begin{cases} 0 & \text{if } (t, c, i) \in I, \\ 1 & \text{otherwise.} \end{cases}$$

Lower bound $a = (a_1, a_2, a_3)$ and upper bound $b = (b_1, b_2, b_3)$ are given as follows:

$$a_1(c, j) = 0, \qquad a_2(t, c) = r(t, c), \qquad a_3(t, j) = 0,$$

$$b_1(c, j) = 1, \qquad b_2(t, c) = r(t, c), \qquad b_3(t, j) = 1.$$

Then the 3-tuple $\langle M, a, b\rangle$ is a 3MP. If there is a solution A of the 3MP $\langle M, a, b\rangle$, it is easy to check that A satisfies all conditions in Definition 2, that is, a timetable. □

The timetable problem discussed above is a special case of that in Definition 1, because there are no considerations of para-lessons, long-lessons, special rooms, upper bound function g and pre-assignments in A_0. Even if a timetable problem is restricted in such a way, it is unsolvable at present.

If we would intend to treat a general case of timetable problem within a framework of nWMP, the formalization of the problem would be so complicated that we could not find a simple and efficient algorithm. Therefore, we need a simple formalization which every requirement for timetabling can be represented uniformly. In order to do this, we propose a combinatorial problem which is referred to as an *existential problem of a weight-controlled subset* or *WSP* in short. We shall discuss the problem WSP and relationship between WSP and timetable problems in the followings.

4. Existential problem of a weight-controlled subset and some results

Definition 5 (Specification of WSP). An *existential problem of a weight-controlled subset* is specified by a 5-tuple $\langle U, S, \omega, a, b\rangle$ satisfying the following conditions:

(1) U is a finite set which is called the *universe*,
(2) S is a collection of subsets of U, i.e. $S \subset 2^U$, which is called a *condition set*,
(3) ω is a function from $M(U, S) = \{(u, s) : u \in U, s \in S, u \in s\}$ to the set of all positive integers, which is called a *weight function*,
(4) a is a function from S to non-negative integers, which is called a *lower bound function*,
(5) b is a function from S to non-negative integers, which is called an *upper bound function*.

Definition 6 (Solution of WSP). A *solution* of WSP $\langle U, S, \omega, a, b\rangle$ is a subset A of U satisfying the following condition:

$$a(s) \leq \sum_{u \in A \cap s} \omega(u, s) \leq b(s) \quad \text{for any } s \text{ in } S.$$

Fig. 2 gives an example of WSP in the case that $S = \{s_1, s_2, s_3\}$, $U = \{u_1, u_2, \ldots, u_9\}$, $a(s_1) = 6$, $a(s_2) = 3$, $a(s_3) = 6$, $b(s_1) = 7$, $b(s_2) = 7$ and $b(s_3) = 8$. Each element of S is indicated by the set of all non-zero elements in the corresponding column of the matrix, e.g. $s_1 = \{u_1, u_2, u_3, u_4, u_9\}$. The matrix also indicates a weight function ω, that is, $(u_i, s_j) = (i, j)$-element of the matrix. A solution A is given as last column of the matrix.

A WSP has never been solved completely. If the number of elements in set U

ω U \ S	s_1	s_2	s_3	A
u_1	2	0	0	0
u_2	1	2	0	0
u_3	3	2	0	1
u_4	1	1	4	1
u_5	0	3	0	0
u_6	0	1	2	1
u_7	0	0	2	0
u_8	0	0	1	0
u_9	2	0	1	1
a	6	3	6	
b	7	7	8	

Fig. 2. Example of WSP.

or S is small, we may find solutions exhaustively by a computer. Unfortunately, U and S in almost all of practical cases, however, have relatively many elements. If we can reduce the size of U and/or S without any change of solvability of the problem, we may use the exhaustive algorithm. The following definitions and some results enable us to use the approach.

Definition 7 (Complete subproblem of WSP). WSP $\langle U, S, \omega, a, b\rangle$ is called a *complete subproblem* of a WSP $\langle U', S', \omega', a', b'\rangle$ iff the following conditions are satisfied:

(1) U is a subset of U',
(2) S is a subset of S',
(3) ω' is a restriction of ω on $M(U', S')$, i.e. $\omega' = \omega|_{M(U',S')}$,
(4) If a WSP $\langle U, S, \omega, a, b\rangle$ has a solution then WSP $\langle U', S', \omega', a', b'\rangle$ also has a solution.

If a WSP P is a complete subproblem of a WSP P', then P' is called an *expansion* of P.

Definition 8 (Trivial subset). Let $\langle U, S, \omega, a, b\rangle$ be a WSP and s be an element of S. If $a(s) \leqslant 0$ then it is said that s has no lower bound. If $b(s) \geqslant \sum_{u \in s} \omega(u, s)$ then it is said that s has no upper bound. If $a(s) \leqslant 0$ and $b(s) \geqslant \sum_{u \in s} \omega(u, s)$ then s is called a *trivial subset.*

Proposition 1 (Reduction of trivial subset). *Let $\langle U, S, \omega, a, b\rangle$ be a WSP and s be a trivial subset. A WSP $\langle U, S', \omega', a', b'\rangle$ that is derived by the removal of s from the WSP $\langle U, S, \omega, a, b\rangle$ is a complete subproblem, where $S' = S - \{s\}$, $\omega' = \omega|_{M(U,S')}$, $a' = a|_{S'}$* and *$b' = b|_{S'}$.*

Since a proof of Proposition 1 is trivial, we shall show only an example that includes a subset s_1 without lower bound, a subset s_2 without upper bound and a trivial subset s_3 in Fig. 3.

ω / U \ S	s_1	s_2	s_3
u_1	1	1	1
u_2	2	1	3
u_3	1	0	0
u_4	0	2	1
a	0	2	0
b	2	5	5

Fig. 3. Example of WSP including a trivial subset.

Proposition 2 (Expansion by intersection). *Let $\langle U, S, \omega, a, b\rangle$ be a WSP and $s_1, \ldots, s_k$ be k elements of S. A WSP $\langle U, S', \omega', a', b'\rangle$ that is derived by intersection of subsets is an expansion of the WSP $\langle U, S, \omega, a, b\rangle$ where*

$$S' = S \cup \{s'\},$$

$$\omega'(u, s) = \begin{cases} \omega(u, s) \text{ if } s \in S \text{ and } u \in s, \\ \max\{\omega(u, s_i): u \in s_i,\ i = 1, 2, \ldots, k\} \text{ if } s = s' \text{ and } u \in s', \end{cases} \tag{4.1}$$

$$a'(s) = \begin{cases} a(s) \text{ if } s \in S, \\ \max(\{a(s_i) - \sum_{u \in (s_i - s')} \omega(u, s_i): i = 1, \ldots, k\} \cup \{0\}) \text{ if } s = s', \end{cases} \tag{4.2}$$

$$b'(s) = \begin{cases} b(s) \text{ if } s \in S, \\ \min\Big(\Big\{b(s_i) + \sum_{u \in s'} (\max\{\omega(u, s_j): j = 1, 2, \ldots, k\} - \omega(u, s_i)): \\ \qquad i = 1, 2, \ldots, k\Big\} \cup \Big\{\sum_{u \in s'} \omega(u, s_i)\Big\}\Big) \text{ if } s = s', \end{cases} \tag{4.3}$$

where $s' = \bigcap_{i=1}^{k} s_i$.

Proof. The following equality is a direct sum decomposition of set s_i.

$$s_i = (s_i - s') \oplus s' \tag{4.4}$$

Let A be a solution of WSP $\langle U, S, \omega, a, b\rangle$. From (4.4), we have

$$\begin{aligned} a(s_i) &\leq \sum_{u \in A \cap s_i} \omega(u, s_i) \\ &= \sum_{u \in A \cap (s_i - s')} \omega(u, s_i) + \sum_{u \in A \cap s'} \omega(u, s_i). \end{aligned} \tag{4.5}$$

On the other hand, for any $i = 1, 2, \ldots, k$,

$$\begin{aligned} & a(s_i) - \sum_{u \in (s_i - s')} \omega(u, s_i) \\ & \leq a(s_i) - \sum_{u \in A \cap (s_i - s')} \omega(u, s_i) \end{aligned}$$

$$\leq \sum_{u\in A\cap s'} \omega(u, s_i) \qquad \text{by (4.5)}$$
$$\leq \sum_{u\in A\cap s'} \max\{\omega(u, s_i): i=1, 2, \ldots, k\}$$
$$= \sum_{u\in A\cap s'} \omega'(u, s') \qquad \text{by (4.1)}.$$

Therefore, we have

$$a'(s') = \max\Big(\Big\{a(s_i) - \sum_{u\in(s_i-s')} \omega(u, s_i): i=1, 2, \ldots, k\Big\} \cup \{0\}\Big)$$
$$\leq \sum_{u\in A\cap s'} \omega'(u, s').$$

On the function b', we can also get the condition in the following: From (4.4) and the definition of WSP, we have

$$\sum_{u\in A\cap s'} \omega(u, s_i) = \sum_{u\in A\cap s_i} \omega(u, s_i) - \sum_{u\in A\cap(s_i-s')} \omega(u, s_i)$$
$$\leq \sum_{u\in A\cap s_i} \omega(u, s_i)$$
$$\leq b(s_i). \tag{4.6}$$

By adding $\sum_{u\in A\cap s'} (\max\{\omega(u, s_j): j=1, 2, \ldots, k\} - \omega(u, s_i))$ to both most left and right sides of inequality (4.6),

$$b(s_i) + \sum_{u\in A\cap s'} (\max\{\omega(u, s_j): j=1, 2, \ldots, k\} - \omega(u, s_i))$$
$$\geq \sum_{u\in A\cap s'} \omega(u, s_i) + \sum_{u\in A\cap s'} (\max\{\omega(u, s_j): j=1, 2, \ldots, k\} - \omega(u, s_i))$$
$$= \sum_{u\in A\cap s'} (\max\{\omega(u, s_j): j=1, 2, \ldots, k\})$$
$$= \sum_{u\in A\cap s'} \omega'(u, s')$$

for any $i=1, 2, \ldots, k$.

Hence we get the inequality that a function b' should be satisfied. □

As an application of Proposition 2, we can give the following corollaries.

Corollary 2.1. *If the set S includes an element s which can be represented by a combination of some other elements in S with intersections and has less strict bounds than that generated by the combination, removal of s from S makes no effect on solvability of the problem.*

Corollary 2.2. *A WSP $\langle U, S, \omega, a, b\rangle$ has a solution A iff the closure of the WSP has a solution A, where the closure of a WSP $\langle U, S, \omega, a, b\rangle$ is a WSP*

$\langle U, S', \omega', a', b' \rangle$ as follows:

(1) *S' is the closure set of S by intersection, i.e. S' satisfies the following conditions:*

  (1) *S is a subset of S',*

  (2) *If s and s' are in S' then an intersection of s and s' is also in S',*

(2) *ω', a' and b' are expansions of ω, a and b respectively in the way of Proposition 2.*

Definition 9 (Inconsistency). A WSP $\langle U, S, \omega, a, b \rangle$ is said to be *inconsistent* if there is an element s in S' that $a'(s) > b'(s)$, where $\langle U, S', \omega', a', b' \rangle$ is the closure of $\langle U, S, \omega, a, b \rangle$.

We shall give an example of inconsistent WSP in Fig. 4. By column-wise investigations, the problem seems to have a solution, but we can recognize that it has no solution by making a new column $s_1 \cap s_2$.

ω S / U	s_1	s_2	$s_1 \cap s_2$
u_1	1	0	0
u_2	3	2	3
u_3	2	2	2
u_4	1	1	1
u_5	0	2	0
a	6	1	5
b	6	3	4

Fig. 4. Example of an inconsistent WSP.

Corollary 2.3. *If a WSP $\langle U, S, \omega, a, b \rangle$ is inconsistent then it has no solution.*

The authors conjecture that the converse of Corollary 2.3 is also true, i.e.

If a WSP has no solution then it is inconsistent,

but we have never solved it.

We have discussed about reduction of columns above and now we shall show how to reduce rows of the matrix $M(U, S)$.

Definition 10 (Deterministic element). Let u be an element in the universe set U of a WSP $\langle U, S, \omega, a, b \rangle$. u is said to be a *deterministic element* when the following condition is satisfied:

There are no solutions A and A' such that u is in A and u is not in A'.

Definition 11 (Dependent element). Let u and u' be elements in the universe set U of a WSP $\langle U, S, \omega, a, b \rangle$. u is said to be *positively* (*negatively*) *dependent* on u'

ω	S	s_1	s_2	s_3	A_1	A_2
U						
u_1		2	1	0	1	0
u_2		1	0	1	1	0
u_3		1	1	2	0	1
u_4		1	2	0	0	0
a		1	1	1		
b		3	1	2		

Fig. 5. Example of deterministic elements and dependent elements.

when the following condition is satisfied:

$$u \text{ is in } A \text{ iff } u' \text{ is (not) in } A \text{ for any solution } A \text{ of } \langle U, S, \omega, a, b\rangle.$$

In Fig. 5, u_1 is positively dependent on u_2 and negatively dependent on u_3. u_4 is a deterministic element. $\{A_1, A_2\}$ is the set of all possible solutions of the WSP.

Deterministic elements and/or positively dependent elements can be eliminated without effect on the existence of a solution of a WSP. The following two propositions give how to do it.

Proposition 3 (Reduction of deterministic element). *Let $\langle U, S, \omega, a, b\rangle$ be a WSP including a deterministic element u. If a WSP $\langle U', S', \omega', a', b'\rangle$ derived by the removal of u is a complete subproblem of the WSP, where*

$$U' = U - \{u\},$$
$$S' = \{s - \{u\} : s \in S\},$$
$$\omega' = \omega|_{M(U',S')},$$
$$a'(s') = \begin{cases} a(s) \text{ if } s' = s, \\ \max(a(s) - v(u)\cdot\omega(u, s), 0) \text{ if } s' = s - \{u\}, \end{cases}$$
$$b'(s') = \begin{cases} b(s) \text{ if } s' = s, \\ \min\left(b(s) - v(u)\cdot\omega(u, s), \sum_{u'\in s'} \omega(u', s')\right) \text{ if } s' = s - \{u\}, \end{cases}$$
$$v(u) = \begin{cases} 1 & \text{if there exist a solution } A \text{ such that } A \ni u, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If A' is a solution of $\langle U', S', \omega', a', b'\rangle$, we have

$$a'(s') \leq \sum_{u'\in A'\cap s'} \omega'(u', s') \leq b'(s') \text{ for any } s' \in S'. \tag{4.7}$$

Let A be a set as follows:

$$A = \begin{cases} A' & \text{if } v(u) = 0, \\ A' \cup \{u\} & \text{if } v(u) = 1. \end{cases}$$

If $v(u) = 0$ or $s' = s$, then inequality (4.7) derives the inequality as follows:

$$a(s) \leqslant \sum_{u' \in A \cap s} \omega(u', s) \leqslant b(s) \text{ for any } s \in S'. \tag{4.8}$$

If $v(u) = 1$ and $s' = s - \{u\}$, then inequality (4.7) can be reformed as follows:

$$a(s) - \omega(u, s) \leqslant \sum_{u' \in (A - \{u\}) \cap (s - \{u\})} \omega(u', s) \leqslant b(s) - \omega(u, s). \tag{4.9}$$

By adding $\omega(u, s)$ to each term of inequality (4.9), we have

$$a(s) \leqslant \sum_{u' \in A \cap s} \omega(u', s) \leqslant b(s). \tag{4.10}$$

Then A is a solution of the WSP $\langle U, S, \omega, a, b \rangle$. □

Proposition 4 (Reduction of positively dependent element). *Let $\langle U, S, \omega, a, b \rangle$ be a WSP and u and u' be elements of U. If u is positively dependent on u' and a WSP $\langle U', S', \omega', a', b' \rangle$ derived by the removal of u' is a complete subproblem of the WSP, where*

$$U' = U - \{u'\},$$
$$S' = \{s - \{u'\} : s \in S\},$$
$$\omega'(u'', s') = \begin{cases} \omega(u'', s) & \text{if } u'' \neq u, \\ \omega(u, s) + \omega(u', s) & \text{if } u'' = u, \end{cases}$$
$$a'(s') = a(s),$$
$$b'(s') = b(s).$$

Proof. If A' is a solution of $\langle U', S', \omega', a', b' \rangle$, we have

$$a'(s') \leqslant \sum_{u'' \in A' \cap s'} \omega'(u'', s') \text{ for any } s' \in S. \tag{4.11}$$

Let A be a set as follows:

$$A = \begin{cases} A' \cup \{u'\} & \text{if } A' \ni u, \\ A' & \text{otherwise.} \end{cases}$$

If $A' \ni u$ then, inequality (4.11) can be reformed step by step as follows:

$$a(s) \leqslant \sum_{u'' \in (A - \{u'\}) \cap (s - \{u'\})} \omega'(u'', s) \leqslant b(s),$$

$$a(s) \leqslant \sum_{u'' \in A \cap s} \omega(u'', s) + \omega(u', s) - \omega(u', s) \leqslant b(s),$$

$$a(s) \leqslant \sum_{u'' \in A \cap s} \omega(u'', s) \leqslant b(s).$$

ω / U	S	s_1	s_2	s_3	A_1	A_2
u_1		3	1	1	1	0
u_3		1	1	2	0	1
a		1	1	1		
b		3	1	2		

Fig. 6. Result by application of Propositions 3 and 4 to a WSP in Fig. 5.

If $A' \not\ni u$ then, from inequality (4.11) we have

$$a(s) \leq \sum_{u'' \in A \cap s} \omega(u'', s) \leq b(s).$$

Then A is a solution of the WSP $\langle U, S, \omega, a, b \rangle$. $\square$

The reduced WSP corresponding to the WSP given in Fig. 5 is given in Fig. 6.

5. Relationship among a timetable problem and a WSP

In this section, we shall show that any timetable problem can be embedded into a WSP. It means that a WSP is more general than a timetable problem. Although both problems have never been solved, a WSP is simpler than a timetable problem.

In order to show that a timetable problem $\langle T, C, W, p, E, L, g, h, r, I, A_0 \rangle$ can be embedded into a WSP, we shall make a WSP $\langle U, S, \omega, a, b \rangle$ from the timetable problem in Lemma 1 and show that a timetable can be constructed from a solution of the WSP in Lemma 2.

Lemma 1. *A WSP can be constructed from a timetable problem.*

Proof. Let $\langle T, C, W, p, E, L, g, h, r, I, A_0 \rangle$ be a timetable problem. Sets U and S, and functions ω, a and b are defined by the following formulas:

$$\begin{aligned} U = {} & L \times W \times \{1, 2, \ldots\} \\ & - \{(t, c, e, n, w, i) : (t, c, e, w, i) \in I, t \in t, c \in c, e \in e\} \\ & - \{(t, c, e, n, w, i) : n + i - 1 > p(w)\}, \end{aligned} \tag{5.1}$$

$$S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6, \tag{5.2}$$

$$\omega(u, s_i) = \begin{cases} 1 & \text{if } i \in \{1, 2, 4, 5, 6\} \text{ and } u \in s_i, \\ n & \text{if } i = 3 \text{ and } u \in s_3, \end{cases}$$

	a	b
$s_1(t, w, j)$	0	1
$s_2(c, w, j)$	0	1
$s_3(t, w)$	0	$g(t, w)$
$s_4(e, w, j)$	0	$h(e)$
$s_5(l)$	$r(l)$	$r(l)$
s_6	1	1

where
$S_1 = \{s_1(t, w, j) : t \in T,\ w \in W,\ j \in \{1, 2, \ldots, p(w)\}\}$,
$S_2 = \{s_2(c, w, j) : c \in C,\ w \in W,\ j \in \{1, 2, \ldots, p(w)\}\}$,
$S_3 = \{s_3(t, w) : t \in T,\ w \in W\}$,
$S_4 = \{s_4(e, w, j) : e \in E,\ w \in W,\ j \in \{1, 2, \ldots, p(w)\}\}$,
$S_5 = \{s_5(l) : l \in L\}$,
$S_6 = \{s_6(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) : (\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in A_0\}$,
$s_1(t, w, j) = \{(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in U : t \in \boldsymbol{t},\ j \in \{i, i+1, \ldots, i+n-1\}\}$,
$s_2(c, w, j) = \{(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in U : c \in \boldsymbol{c},\ j \in \{i, i+1, \ldots, i+n-1\}\}$,
$s_3(t, w) = \{(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in U : t \in \boldsymbol{t}\}$,
$s_4(e, w, j) = \{(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in U : e \in \boldsymbol{e},\ j \in \{i, i+1, \ldots, i+n-1\}\}$,
$s_5(l) = \{(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i) \in U : l = (\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n),\ w \in W,\ i \in \{1, 2, \ldots, p(w)\}\}$,
$s_6(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i\} = \{(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i)\}$.

It is obvious that the 5-typle $\langle U, \boldsymbol{S}, \omega, \boldsymbol{a}, b\rangle$ is a WSP. □

Lemma 2. *If there is a solution of a WSP derived from a timetable problem in the way of Lemma* 1, *a solution of the timetable problem can be constructed.*

Proof. Let $\langle U, \boldsymbol{S}, \omega, a, b\rangle$ be a WSP which is derived from a timetable problem $\langle T, C, W, p, E, L, g, h, r, I, A_0\rangle$ by Lemma 1 and A be a solution of the WSP. We shall show that the set A is also a solution of the timetable problem. In order to do this, we shall check that A satisfies the condition (1), (2), ..., (8) in Definition 2.

Condition (1) and (7) is obvious from the definition of A.

Condition (2): If two difference elements $(\boldsymbol{t}, \boldsymbol{c}, \boldsymbol{e}, n, w, i)$ and $(\boldsymbol{t}', \boldsymbol{c}', \boldsymbol{e}', n', w, i')$ of A satisfy the following two conditions simultaneously:

$$\boldsymbol{t} \cap \boldsymbol{t}' \ni t,$$

$$\{i, i+1, \ldots, i+n-1\} \cap \{i', i'+1, \ldots, i'+n'-1\} \ni j,$$

then set $s_1(t, w, j)$ includes the two elements in A. Then we have

$$|A \cap s_1(t, w, j)| \geqslant 2. \tag{5.3}$$

On the other hand, we have the following inequality:

$$|A \cap s_1(t, w, j)| = \sum_{u \in A \cap s_1(t, w, j)} \omega(u, s_1) \leqslant b(s_1(t, w, j)) = 1,$$

because A is a solution of the WSP and $\omega(u, s_1) = 1$ when u is in s_1. This is contradiction to (5.3) and then the set A satisfies the Condition (2).

Condition (3) can be derived in the same way as Condition (2).

Condition (4) can be derived immediately from the inequality of a solution A and the definition of weight function ω by checking the following equalities:

$$A \cap s_3(t, w) = \{(t, c, e, n, w, i) \in A : t \in t\},$$

$$\omega(u, s_3) = n \text{ if } u \in s_3.$$

Conditions (5) and (6) can also be derived immediately from the inequality of a solution A, the definitions of subsets s_4 and s_5 and the definition of weight function ω in the same way as Condition (3).

Condition (8): From the inequality of a solution A, we have

$$1 = a(s) \leq \sum_{u \in A \cap s_6} \omega(u, s) \leq b(s) = 1 \text{ for any } s \text{ in } S_6.$$

On the other hand, $\omega(u, s_6) = 1$ if u is in s_6 and $|s_6| = 1$. Therefore we get $A \supset s_6$, that is $A \supset A_0$. □

By Lemma 1 and Lemma 2, we can get the main theorem.

Theorem 2. *A timetable problem can be embedded into a WSP.*

6. Conclusion and discussion

A school timetable problem is a very important and interesting problem. The quality of a timetable has a great impact upon the effects of education for pupils/students and labor condition of teachers. From the educational point of view, a timetable is one of the most important environments of education as well as quality and quantity of teachers, equipments, curriculums and so on. In spite of the importance, it has not been used long enough to keep the quality of timetable good, because timetable construction must be done in a very tight schedule. All requirements for a timetable are usually fixed in one or two weeks before the beginning of a new term. This means that there is not enough time for timetable construction manually. The larger school has more difficulty since the time needed for constructing a timetable depends on the size of school.

In order to overcome the difficulties, there have been many trials to construct a timetable by computers. They derived the theory of school timetables which intends to establish a simple and powerful model of timetable and find efficient algorithms for constructing timetables and checking the constrictivity. Among models and algorithms proposed in advance, some of them restrict conditions too strong to accept the timetable for actual schools, and others can support only trial-and-error manual operations by a computer.

In this paper, we proposed a new theoretical model for school timetable construction, which can represent various complex requirements which frequently appeared in the practical timetable construction. These have been treated as exceptional conditions in the models developed in advance. In spite of powerful

representability of our new model, it seems very simple. Unfortunately, it has never been solved completely. This paper gives some theoretical results that are useful for establishing the efficient algorithms.

The combinatorial problem derived from a timetable problem is not only important in the practical sense, but also interesting in the theoretical sense. The problem is a generalized one of various classical problems in graph theory, e.g. coloring problem, Latin square problem, network flow problem and resource dispatching problem. Full-scale theoretical attack to the problem is our future plan.

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MORE NON-RECONSTRUCTIBLE HYPERGRAPHS

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A *k-hypergraph* G consists of a *vertex-set* $V(G)$ and an *edge-set* $E(G)$, a set of k-subsets of $V(G)$. If $X \subseteq V(G)$, the edges of the induced subgraph $G[X]$ are those edges of G whose vertices are all contained in X. If $v \in V(G)$, then $G - v$ denotes the induced subgraph $G[V(G) - v]$. Hypergraphs G and H are *hypomorphic* if there is a bijection $\phi : V(G) \to V(H)$ such that $G - v \cong H - \phi(v)$, for all $v \in V(G)$. G and H are said to be *reconstructions* of each other. In case $G \ncong H$, G and H are said to be *non-reconstructible*. We describe the construction of a family of non-reconstructible 3-hypergraphs with $2^n + 2^m$ vertices, for all $n, m \geq 1$.

1. Introduction

Let $k > 1$ be an integer. Let V be any set with $|V| \geq k$. We denote by $\binom{V}{k}$ the set of all k-subsets of V. A *k-hypergraph* G consists of a set $V(G)$ of *vertices* and a set $E(G) \subseteq \binom{V(G)}{k}$ of *edges*. If $X \subseteq V(G)$, then the subhypergraph *induced* by X is $G[X] \equiv \{e \in E(G) \mid e \in \binom{X}{k}\}$. The *vertex-deleted* hypergraph $G - u$ is defined to be $G[V(G) - \{u\}]$, where $u \in V(G)$.

Two k-hypergraphs G and H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $\theta : V(G) \to V(H)$ such that $e \in E(G)$ if and only if $\theta(e) \in E(H)$. G and H are *hypomorphic* if there is a bijection $\phi : V(G) \to V(H)$ such that $G - u \cong H - \phi(u)$, for each $e \in V(G)$.

If $k = 2$ the *graph reconstruction conjecture* [2] *states that if* $|V(G)| \geq 3$, then if G and H are hypomorphic, then they are also isomorphic. Hypomorphic graphs are said to be *reconstructions* of each other. If $G \ncong H$, then G and H are *non-reconstructible*. The corresponding question for k-hypergraphs is: if $|V(G)| > k \geq 3$, then if G and H are hypomorphic k-hypergraphs, is $G \cong H$? In [2] it was shown that this statement can be false for 3-hypergraphs, and that it is most likely false for all $k > 2$. (A 2-hypergraph is a graph in the usual sense, with no repeated edges.) In [2] a family of non-isomorphic, hypomorphic pairs of 3-hypergraphs with $2^n + 1$ vertices was constructed, for all $n \geq 1$. In this note, we generalize the construction of [2], constructing a family of non-reconstructible 3-hypergraphs with $2^n + 2^m$ vertices, for all values of $n, m \geq 0$.

We consider only 3-hypergraphs from now on, so that the term "hypergraph" will always refer to a 3-hypergraph. The construction to be presented requires the

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hypergraphs $G^{\varepsilon}_{j,k}(n)$ of [2]. We begin by defining them and stating several of their properties proved in [2].

2. The families $G^{\varepsilon}_{j,k}(n)$ and $A^{\varepsilon}_{j}(n)$

Let $n \geq 0$ be an integer. Define $V_n = \{1, 2, \ldots, 2^n\}$. Define two maps p_{odd}, $p_{\text{even}}: \mathbf{Z} \to \mathbf{Z}$ by $p_{\text{odd}}(k) = 2k - 1$ and $p_{\text{even}}(k) = 2k$. We consider all maps of integers to act also on objects composed of integers, so that $p_{\text{odd}}(V_n) \cup p_{\text{even}}(V_n) = V_{n+1}$, for any $n \geq 0$. similarly, we shall allow p_{odd} and p_{even} to act on graphs and hypergraphs via their edge and vertex sets. If $x, y \in V_n$, we compute the sum $x + y \bmod V_n$ as the unique integer $z \in V_n$ such that $x + y \equiv z \pmod{2^n}$.

If G and H are hypergraphs such that $E(G) \cap E(H) = \emptyset$, we write $G + H$ for the hypergraph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $E(G) \cup E(H)$. If G is a hypergraph with vertex set V_{n-1}, we define a new hypergraph, denoted Eight(G), with vertex-set V_n, by replacing each edge of G by an eightfold copy, as follows. Let $N = 2^{n-1}$. If $\{x, y, z\} \in E(G)$, then Eight(G) contains the edges

$$\begin{array}{llll} \{x, y, z\}, & \{x, y, z+N\}, & \{x, y+N, z\}, & \{x+N, y, z\}, \\ \{x, y+N, z+N\}, & \{x+N, y, z+N\}, & & \\ \{x+N, y+N, z\}, & \{x+N, y+N, z+N\}, & & \end{array}$$

where the sums are computed mod V_n.

Definition 2.1. Let $1 \leq j < k \leq n \geq 2$, and let $\varepsilon \in \{0, 1\}$. We define a family $G^{\varepsilon}_{j,k}(n)$ of hypergraphs with vertex set V_n.

(1) If $n = 2$, then

$$E(G^0_{1,2}(2)) = \{\{1, 2, 3\}, \{2, 3, 4\}\}$$
$$E(G^1_{1,2}(2)) = \{\{1, 2, 4\}, \{1, 3, 4\}\}.$$

(2) If $n = 3$ and $(j, k) = (1, 3)$ then

$$E(G^0_{1,3}(3)) = \{\{x, x+1, x+2\}, \{x, x+3, x+5\} \mid x \in V_3\}$$
$$E(G^1_{1,3}(3)) = \{\{x, x+1, x+3\}, \{x, x+5, x+7\} \mid x \in V_3\},$$

where the addition is computed mod V_3.

(3) If $n \geq 3$ and $k < n$ then

$$G^{\varepsilon}_{j,k}(n) = p_{\text{odd}}(G^{\varepsilon}_{j,k}(n-1)) + p_{\text{even}}(G^{\varepsilon}_{j,k}(n-1)),$$

i.e. make two copies of $G^{\varepsilon}_{j,k}(n-1)$, one odd and one even, and take their union.

(4) If $n \geqslant 3$, $k = n$, and $j = n - 1$, then

$$E(G^0_{n-1,n}(n)) = \{\{y, y+N, 4x+y-3\} \mid y \in V_n, x \in V_{n-3}\},$$
$$E(G^1_{n-1,n(n))} = \{\{y, y+N, 4x+y-1\} \mid y \in V_n, x \in V_{n-3}\},$$

where $N = 2^n - 1$ and addition is computed mod V_n.

(5) If $n \geqslant 4$, $k = n$, and $j < n - 1$, then

$$G^\varepsilon_{j,n}(n) = \mathrm{Eight}(G^\varepsilon_{j,n-1}(n-1)),$$

i.e., make an eightfold copy of each edge of $G^\varepsilon_{j,n-1}(n-1)$.

It is easily checked that this defines a unique hypergraph for each value of (ε, j, k), so that $n(n-1)$ hypergraphs have been defined. We state without proof a number of properties of the $G^\varepsilon_{j,k}(n)$.

2.2. $G^\varepsilon_{j,k}(n)$ and $G^\delta_{i,m}(n)$ have no edges in common if $(\delta, i, m) \neq (\varepsilon, j, k)$.
2.3. $G^\varepsilon_{j,k}(n)$ partition $\binom{V_n}{3}$, where $1 \leqslant j < k \leqslant n \geqslant 2$.
2.4. $G^0_{j,k}(n) \cong G^1_{j,k}(n)$.
2.5. Let $\theta_n : V_n \to V_n$ denote the map $x \to 2^n - x + 1$, where $x \in V_n$. If $n \geqslant 3$ and $j \neq 1$, then $\theta_n(G^\varepsilon_{j,k}(n)) = G^{1-\varepsilon}_{j,k}(n)$.
2.6. Let $n \geqslant 2$. Then $\theta_n(G^\varepsilon_{1,k}(n)) = G^\varepsilon_{1,k}(n)$.

We also define a family $A^\varepsilon_j(n)$ of graphs (2-hypergraphs) with vertex set V_n, for $1 \leqslant j \leqslant n \geqslant 2$ and $\varepsilon \in \{0, 1\}$.

Definition 2.7.

(1) If $n \geqslant 2$ and $j = 1$, then

$$E(A^0_1(2)) = \{\{1, 3\}\},$$
$$E(A^1_1(2)) = \{\{2, 4\}\}.$$

(2) If $n \geqslant 2$ and $j = n$, then

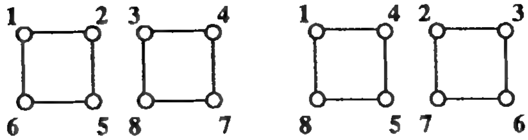
$$E(A^0_n(n)) = \{\{4x-3, 4y-2\}, \{4x-1, 4y\} \mid x, y \in V_{n-2}\}$$
$$E(A^1_n(n)) = \{\{4x-3, 4y\}, \{4x-1, 4y-2\} \mid x, y \in V_{n-2}\}.$$

(3) If $n \geqslant 3$ and $j < n$, then

$$A^0_j(n) = p_{\mathrm{odd}}(A^0_j(n-1) + A^1_j(n-1)),$$
$$A^1_j(n) = p_{\mathrm{even}}(A^0_j(n-1) + A^1_j(n-1)).$$

For example $A^0_3(3)$ and $A^1_3(3)$ are illustrated in Fig. 1.

The graphs $A^0_j(n)$ also satisfy a number of properties which are stated here without proof.



$A_3^0(3)$ $A_3^1(3)$

Fig. 1.

2.8. $A_j^\varepsilon(n)$ and $A_i^\delta(n)$ have no edges in common if $(\delta, i) \neq (\varepsilon, j)$.
2.9. $A_j^\varepsilon(n))$ partition $\binom{V_n}{2}$, where $1 \leqslant j \leqslant n \geqslant 2$, and $\varepsilon \in \{0, 1\}$.
2.10. If $j < n$, then $\theta_n(A_j^\varepsilon(n)) = A_j^{1-\varepsilon}(n)$.
2.11. $\theta_n(A_n^\varepsilon(n)) = A_n^\varepsilon(n)$.
2.12. $A_j^\varepsilon(n) \cong A_j^{1-\varepsilon}(n)$.

We now introduce two sets of indeterminates $\{u_i \mid i \in V_n\}$ and $\{v_i \mid i \in V_n\}$, using the symbols u and v, subscripted by elements of V_n. Denote these two sets by $V_n(u)$ and $V_n(v)$. We shall define families of hypergraphs with vertex-sets $V_n(u)$ and $V_m(v)$, where $n, m \geqslant 0$. Write $G_{j,k}^\varepsilon(n; u)$ for the hypergraph formed from $G_{j,k}^\varepsilon(n)$ by replacing each vertex $i \in V_n$ with $u_i \in V_n(u)$. $G_{j,k}^\varepsilon(n; v)$, $A_j^\varepsilon(n; u)$, and $A_j^\varepsilon(n; v)$ are defined similarly. Call these two sets of hypergraphs u-hypergraphs and v-hypergraphs, according as to whether the vertex-set is $V_n(u)$ or $V_n(v)$.

If A denotes a set of unordered pairs (e.g. the edges of a graph) and u is a vertex, then uA denotes the set of all triples formed by adding u to each pair of A, so that uA is the edge set of a hypergraph.

Definition 2.13. We define hypergraphs $M_j^k(n, m)$ and $N_i^\varepsilon(n, m)$, with vertex-set $V_n(u) \cup V_m(v)$, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

(1) If $n \geqslant 1$ and $m \geqslant 2$, then:

$$E(M_j^0(n, m)) = \{u_{2k-1}E(A_j^0(m; v)) \mid k \in V_{n-1}\},$$
$$E(M_j^1(n, m)) = \{u_{2k}E(A_j^1(m; v)) \mid k \in V_{n-1}\};$$

(2) If $n \geqslant 2$ and $m \geqslant 1$, then:

$$E(N_i^0(n, m)) = \{v_{2k-1}E(A_i^0(n; u)) \mid k \in V_{m-1}\},$$
$$E(N_i^1(n, m)) = \{v_{2k}E(A_i^1(n; u)) \mid k \in V_{m-1}\};$$

(3) If $n = 0$ and $m \geqslant 2$, then:

$$E(M_j^0(0, m)) = u_1E(A_j^0(m; v)),$$
$$E(M_j^1(0, m)) = u_1E(A_j^1(m; v));$$

(4) If $n \geqslant 2$ and $m = 0$, then:

$$E(N_i^0(n, 0)) = v_1E(A_i^0(n; u)),$$
$$E(N_i^1(n, 0)) = v_1E(A_i^1(n; u)).$$

Each hypergraph $G^{\varepsilon}_{j,k}(n;u)$ or $G^{\varepsilon}_{j,k}(m;v)$ has vertex-set $V_n(u)$ or $V_m(v)$, respectively. They are called *pure* hypergraphs. Each edge of $M^{\varepsilon}_i(n,m)$ contains one vertex of $V_n(u)$ and two vertices of $V_m(v)$. Each edge of $N^{\varepsilon}_i(n,m)$ contains one vertex of $V_m(v)$ and two of $V_n(u)$. They are called *mixed* hypergraphs.

If G is a hypergraph and $u \in V(G)$, the *degree* of u is $\deg(u, G)$, the number of edges of G containing u. The following results are easily verified from the definition or else can be found in [2].

2.14. Let $i \in V_n$, $n \geqslant 3$, and $(j,k) \neq (1,2)$. Then $\deg(i, G^{\varepsilon}_{j,k}(n)) = 3 \cdot 2^{2k-j+4}$.

2.15. Let $i \in V_n$, $n \geqslant 3$. Then

$$\deg(i, G^{\varepsilon}_{1,2}(n)) = \begin{cases} 1+\varepsilon, & \text{if } i \leqslant 2^{n-2} \text{ or } i > 3\cdot 2^{n-2}, \\ 2-\varepsilon, & \text{if } 2^{n-2} < i \leqslant 3 \cdot 2^{n-2}. \end{cases}$$

2.16. Let $1 \leqslant j \leqslant n$ and $i \in V_n$, where $n \geqslant 2$. Then $|E(A^{\varepsilon}_j(n))| = 2^{n+j-3}$ and $\deg(i, A^{\varepsilon}_j(n)) = 2^{j-3}$.

By forming hypergraphs of the form $G^{\varepsilon}_{j,k}(n;u) + M^{\delta}_i(n,m) + N^{\eta}_h(n,m) + G^{\zeta}_{t,s}(m;v)$, we shall construct families of non-reconstructible hypergraphs with vertex-set $V_n(u) \cup V_m(v)$. We first need some information about the isomorphisms between the vertex-deleted hypergraphs.

3. The mappings $P_{n,i}$

Definition 3.1. Let $n \geqslant 1$ and let $i \in V_n$. We define (see [2]) a family of mappings $P_{n,i}\colon V_n - i \to V_n - i$.

(1) If $n = 1$ then $P_{1,1}(2) = 2$ and $P_{1,2}(1) = 1$.

(2) If $i = 2j$, $j \neq x \in V_{n-1}$, and $n \geqslant 2$, then

$$P_{n,i}(2x-1) = 2^n - 2x + 1,$$
$$P_{n,i}(2x) = 2P_{n-1,j}(x).$$

(3) If $i = 2j-1$, $j \neq x \in V_{n-1}$, and $n \geqslant 2$, then

$$P_{n,i}(2x-1) = 2P_{n-1,j}(x) - 1,$$
$$P_{n,i}(2x) = 2^n - 2x + 2.$$

Proofs of the following properties can be found in [2].

3.2. Let $n \geqslant 2$ and $j \neq 1$. Then $P_{n,i}(G^{\varepsilon}_{j,k}(n) - i) = G^{1-\varepsilon}_{j,k}(n) - i$, where $i \in V_n$.

3.3. Let $1 < k \leqslant n \geqslant 2$. Then $P_{n,i}(G^{\varepsilon}_{1,k}(n) - i) = G^{\varepsilon}_{1,k}(n) - i$, where $i \in V_n$.

3.4. Let $n \geqslant 2$. Then $P_{n,i}(A^{\varepsilon}_n(n) - i) = A^{1-\varepsilon}_n(n) - i$, where $i \in V_n$.

3.5. Let $1 \leqslant j < n \geqslant 2$. Then $P_{n,i}(A^{\varepsilon}_j(n) - i) = A^{\varepsilon}_j(n) - i$, where $i \in V_n$.

3.6. $P_{n,i}$ maps odd numbers to odd and even to even.

The significance of these lemmas is the following. Associated with each hypergraph $G^{\varepsilon}_{j,k}(n)$ is a *hypomorph* $\mathrm{Hypo}(G^{\varepsilon}_{j,k}(n))$, and associated with each graph $A^{\varepsilon}_{j}(n)$ is a hypomorph $\mathrm{Hypo}(A^{\varepsilon}_{j}(n))$:

$$\mathrm{Hypo}(G^{\varepsilon}_{j,k}(n)) = G^{1-\varepsilon}_{j,k}(n), \quad \text{if } j \neq 1;$$
$$\mathrm{Hypo}(G^{\varepsilon}_{1,k}(n)) = G^{\varepsilon}_{1,k}(n), \quad \text{if } j = 1;$$
$$\mathrm{Hypo}(A^{\varepsilon}_{j}(n)) = A^{\varepsilon}_{j}(n), \quad \text{if } j < n;$$
$$\mathrm{Hypo}(A^{\varepsilon}_{n}(n)) = A^{1-\varepsilon}_{n}(n), \quad \text{if } j = n.$$

In all cases $P_{n,i}(G^{\varepsilon}_{j,k}(n) - i) = \mathrm{Hypo}(G^{\varepsilon}_{j,k}(n)) - i$, and $P_{n,i}(A^{\varepsilon}_{j}(n) - i) = \mathrm{Hypo}(A^{\varepsilon}_{j}(n)) - i$, so that the *same mappings* $P_{n,i}$ map a vertex deleted graph or hypergraph to the vertex-deleted graph or hypergraph of its hypomorph. Because of this, *we are free to take any combination of these hypergraphs and the resulting hypergraph will still have a well-defined hypomorph for which the mappings* $P_{n,i}$ *will map the vertex-deleted hypergraphs to the corresponding vertex-deleted hypergraphs of the hypomorph.*

We extend θ_n to act on $V_n(u)$ and $V_n(v)$ by writing $\theta_n(u_i) = u_{\theta_n(i)}$ and $\theta_n(v_i) = v_{\theta_n(i)}$.

Definition 3.7. We also extend θ_n and the $P_{n,i}$ to act on the set $V_n(u) \cup V_m(v)$ as follows: Let $i \in V_n$ and $j \in V_m$. Then

(1) $\theta^m_n(u_i) = \theta_n(u_i)$ and $\theta^m_n(v_j) = \theta_m(v_j)$;

(2) $P^m_{n,i}(v_j) = \theta_m(v_j)$ for all $v_j \in V_m(v)$,
$P^m_{n,i}(u_k) = u_t$ if $P_{n,i}(k) = t$;

(3) $P^{m,j}_n(u_i) = \theta_n(u_i)$ for all $u_i \in V_n(u)$.
$P^{m,j}_n(v_k) = v_t$ if $P_{m,j}(k) = t$;

i.e. the superscript m refers to $V_m(v)$ and the subscript n to $V_n(u)$.

In this way, the $P^m_{n,i}$ and $P^{m,i}_n$ will also be isomorphisms of the vertex-deleted hypergraphs of $G^{\varepsilon}_{j,k}(n;u)$ and $G^{\varepsilon}_{j,k}(n;v)$ with those of their hypomorphs.

Lemma 3.8. *Let* $1 \leq j < m$ *and* $1 \leq i < n$. *Then*:

$$\theta^m_n(M^{\varepsilon}_{j}(n,m)) = M^{\varepsilon}_{j}(n,m),$$
$$\theta^m_n(M^{\varepsilon}_{m}(n,m)) = M^{1-\varepsilon}_{m}(n,m),$$
$$\theta^m_n(N^{\varepsilon}_{i}(n,m)) = N^{\varepsilon}_{i}(n,m), \quad \text{and}$$
$$\theta^m_n(N^{\varepsilon}_{n}(n,m)) = N^{1-\varepsilon}_{n}(n,m).$$

Proof. Consider $M^{\varepsilon}_{j}(n,m)$. By 2.13, the edges of $M^{\varepsilon}_{j}(n,m)$ containing any u_{2k-1}

form a copy of $A_j^0(m;v)$, and the edges containing any u_{2k} form a copy of $A_j^1(m;v)$. But $\theta_n^m(A_j^\varepsilon(m;v)) = A_j^{1-\varepsilon}(m;v)$ by 2.10 and 3.7. Furthermore, θ_n maps odd to even and even to odd, so that $\theta_n^m(M_j^\varepsilon(n,m)) = M_j^\varepsilon(n,m)$. The other results follow similarly, using 2.10, 2.11, and 2.13. □
(Note: This result does hold when $n = 0$ or $m = 0$ is allowed.)

Lemma 3.9. *Let $i \in V_n$, $k \in V_m$, and $1 \leq i < m$. Then $P_{n,i}^m(M_j^\varepsilon(n,m) - u_i) = M_j^{1-\varepsilon}(n,m) - u_i$, and $P_n^{m,k}(M_j^\varepsilon(n,m) - v_k) = M_j^{1-\varepsilon}(n,m) - v_k$.*

Proof. Consider $P_{n,i}^m(M_j^\varepsilon(n,m) - u_i)$, where $i \in V_n$. $P_{n,i}^m$ maps each v_k to $\theta_m(v_k)$. By 2.13 the edges of $M_j^\varepsilon(n,m)$ containing any u_k from a copy of $A_j^0(m;v)$ or $A_j^1(m;v)$, and $\theta_m(A_j^\varepsilon(m;v)) = A_j^{1-\varepsilon}(m;v)$, by 2.10. Furthermore, $P_{n,i}$ maps odd to odd and even to even, by 3.7, so that by considering the Definition 2.13, it is easy to see that $P_{n,i}^m(M_j^\varepsilon(n,m) - u_i) = M_j^{1-\varepsilon}(n,m) - u_i$.

Consider now $P_n^{m,k}(M_j^\varepsilon(n,m) - v_k)$. If $i \in V_n$ is odd, then the edges of $M_j^\varepsilon(n,m) - v_k$ containing u_i define a copy of $A_j^0(m;v) - v_k$. $P_n^{m,k}$ will map u_i to $\theta_n(u_i)$. By 3.5, $P_{m,k}(A_j^\varepsilon(m) - k) = A_j^\varepsilon(m) - k$, so that $P_n^{m,k}(A_j^\varepsilon(m;v) - v_k) = A_j^\varepsilon(m;v) - v_k$. But since i is odd, $\theta_n(i)$ is even, so that in $M_j^\varepsilon(n,m) - v_k$, the edges containing $\theta_n(u_i)$ from a copy of $A_j^1(m;v) - v_k$, i.e. in $P_n^{m,k}(M_j^\varepsilon(n,m) - v_k)$ the edges containing each u_i are the same as in $M_j^{1-\varepsilon}(n,m) - v_k$. If i is even, then $\theta_n(i)$ is odd and the analysis is similar, so that $P_{n,i}^m(M_j^\varepsilon(n,m) - u_i) = M_j^{1-\varepsilon}(n,m) - u_i$. □

Lemma 3.10. *Let $i \in V_n$, $k \in V_m$, and $1 \leq j < n$. Then $P_{n,i}^m(N_j^\varepsilon(n,m) - u_i) = N_j^{1-\varepsilon}(n,m) - u_i$, and $P_n^{m,k}(N_j^\varepsilon(n,m) - v_k) = N_j^{1-\varepsilon}(n,m) - v_k$.*

Proof. The proof is like that of 3.9, except that the roles of u and v are reversed. □

Lemma 3.11. *Let $i \in V_n$ and $k \in V_m$. Then $P_{n,i}^m(M_m^\varepsilon(n,m) - u_i) = M_m^\varepsilon(n,m) - u_i$, and $P_n^{m,k}(M_m^\varepsilon(n,m) - v_k) = M_m^\varepsilon(n,m) - v_k$.*

Proof. The proof is very much like that of 3.9. Consider $P_{n,i}^m(M_m^\varepsilon(n,m) - u_i)$, where $i \in V_m$. $P_{n,i}^m$ maps each v_k to $\theta_m(v_k)$. By 2.13 the edges of $M_m^\varepsilon(n,m)$ containing any u_k form a copy of $A_m^0(m;v)$ or $A_m^1(m;v)$, and $\theta_m(A_m^\varepsilon(m;v)) = A_m^\varepsilon(m;v)$, by 2.11. Furthermore, $P_{n,i}$ maps odd to odd and even to even, by 3.7, so that by considering the Definition 2.13, it is easy to see that $P_{n,i}^m(M_m^\varepsilon(n,m) - u_i) = M_m^\varepsilon(n,m) - u_i$.

Consider now $P_n^{m,k}(M_m^\varepsilon(n,m) - v_k)$. If $i \in V_n$ is odd, then the edges of $M_m^\varepsilon(n,m) - v_k$ containing u_i define a copy of $A_m^0(m;v) - v_k$. $P_n^{m,k}$ will map u_i to $\theta_n(u_i)$. By 3.4, $P_{m,k}(A_m^\varepsilon(m) - k) = A_m^{1-\varepsilon}(m) - k$, so that $P_n^{m,k}(A_m^\varepsilon(m;v) - v_k) = A_m^{1-\varepsilon}(m;v) - v_k$. But since i is odd, $\theta_n(i)$ is even, so that in $M_m^\varepsilon(n,m) - v_k$, the edges containing $\theta_n(u_i)$ form a copy of $A_m^1(m;v) - v_k$, i.e. in $P_n^{m,k}(M_m^\varepsilon(n,m) -$

v_k) the edges containing each u_i are the same as in $M_m^\varepsilon(n, m) - v_k$. If i ie even, then $\theta_n(i)$ is odd and the analysis is similar, so that $P_{n,i}^m(M_m^\varepsilon(n, m) - u_i) = M_m^\varepsilon(n, m) - u_i$. □

Lemma 3.12. *Let $i \in V_n$ and $k \in V_m$. Then $P_{n,i}^m(N_n^\varepsilon(n, m) - u_i) = N_n^\varepsilon(n, m) - u_i$, and $P_n^{m,k}(N_n^\varepsilon(n, m) - v_k) = N_n^\varepsilon(n, m) - v_k$.*

Proof. The proof is like that of 3.11, except that the roles of u and v are reversed. □

Lemmas 3.9 to 3.12 show that we can define hypomorphs for the $M_j^\varepsilon(n, m)$ and $N_i^\varepsilon(n, m)$:

$$\mathrm{Hypo}(M_j^\varepsilon(n, m)) = M_j^{1-\varepsilon}(n, m), \quad \text{if } j < m;$$
$$\mathrm{Hypo}(M_m^\varepsilon(n, m)) = M_m^\varepsilon(n, m), \quad \text{if } j = m;$$
$$\mathrm{Hypo}(N_i^\varepsilon(n, m)) = N_i^{1-\varepsilon}(n, m), \quad \text{if } i < n;$$
$$\mathrm{Hypo}(N_n^\varepsilon(n, m)) = N_n^\varepsilon(n, m), \quad \text{if } i = n.$$

In each case the hypomorphism is the identity mapping from $V_n(u) \cup V_m(v)$ to itself, and the mappings $P_{n,i}^m$ and $P_n^{m,k}$ are the isomorphisms from the vertex-deleted hypergraphs to those of the hypomorph. It is clear that the same mappings are also isomorphisms from the vertex-deleted hypergraphs of $G_{j,k}^\varepsilon(n; u)$ and $G_{j,k}^\varepsilon(m; v)$ to those of their hypomorphs, too.

4. Non-reconstructible families

In this section we put the preceding results together and construct a family of non-reconstructible hypergraphs with vertex-set $V_n(u) \cup V_m(v)$, giving $2^n + 2^m$ vertices in total. We first summarize the results of Section 3 in a table for handy reference.

Definition 4.1. Let $n \geq 2$. Then G_n^ε denotes the hypergraph $\sum_{k=2}^n G_{1,k}^\varepsilon(n)$. $G_n^\varepsilon(u)$ and $G_n^\varepsilon(v)$ are defined correspondingly.

The *automorphism group* of a hypergraph G is $\mathrm{Aut}(G)$, the group consisting of all permutations of $V(G)$ which are isomorphisms of G with itself. The following lemma is proved in [2].

Lemma 4.2. *Let $n \geq 3$. Then $Aut(G_n^\varepsilon) = \langle \theta_n \rangle$.*

Theorem 4.3. *Let $n, m \geq 3$, $n \neq m$, and $1 \leq j < m$. Let $X_{j,n,m}^\varepsilon = G_n^\varepsilon(u) + M_j^\varepsilon(n, m) + G_m^\varepsilon(v)$, and let $Y_{j,n,m}^\varepsilon = \mathrm{Hypo}(X_{j,n,m}^\varepsilon) = G_n^\varepsilon(u) + M_j^{1-\varepsilon}(n, m) + G_m^\varepsilon(v)$. Then $X_{j,n,m}^\varepsilon \not\cong Y_{j,n,m}^\varepsilon$.*

Proof. By 2.14 and 2.15 $\deg(u_k, G_n^\varepsilon(u))$ is either 2^{2n-3} or $2^{2n-3}-1$, and $\deg(v_k, G_m^\varepsilon(v))$ is either 2^{2m-3} or $2^{2m-3}-1$. By 2.13 and 2.16, $\deg(u_k, M_j^\varepsilon(n, m)) = 2^{m+j-3}$ and $\deg(v_k, M_j^\varepsilon(n, m)) = 2^{n+j-3}$. So $\deg(u_k, X_{j,n,m}^\varepsilon) = 2^{2n-3} - \delta + 2^{m+j-3}$ and $\deg(v_k, X_{j,n,m}^\varepsilon) = 2^{2m-3} - \zeta + 2^{n+j-3}$, where δ and ζ are either 0 or 1. Since $j < m$ and $n \neq m$, it follows that vertices of $V_n(u)$ and $V_m(v)$ are distinguished by their degree, so that if $\psi: X_{j,n,m}^\varepsilon \to Y_{j,n,m}^\varepsilon$ is any putative isomorphism, ψ must map $V_n(u)$ to $V_n(u)$ and $V_m(v)$ to $V_m(v)$. Since each edge of $G_n^\varepsilon(u)$ contains only vertices of $V_n(u)$, whereas edges of $M_j^\varepsilon(n, m)$ are mixed, it follows that $\psi(G_n^\varepsilon(u)) = G_n^\varepsilon(u)$, or $\psi \in \mathrm{Aut}(G_n^\varepsilon(u))$. The only possible choice is ψ acts as θ_n on $V_n(u)$, by 4.2. Similarly, ψ must act as θ_m on $V_m(v)$, so that $\psi = \theta_n^m$ is the only possibility. But $\theta_n^m(M_j^\varepsilon(n, m)) = M_j^\varepsilon(n, m)$, by Table 1, from which it follows that $X_{j,n,m}^\varepsilon \not\cong Y_{j,n,m}^\varepsilon$. □

For example, $X_{2,3,2}^0$ is illustrated in Fig. 2. Hypergraphs are not so easily drawn on paper as graphs are, where their structure is immediately visible. We have used the following technique of [1] to represent a hypergraph graphically. If G is any hypergraph and $u \in V(G)$, write A_u for the set of edges containing vertex u. If we now remove u from each edge of A_u, we get the edge set of a graph A_u'. So G defines a set $\{A_u' \mid u \in V(G)\}$ of graphs. Following [1], we say that G *subsumes* the graphs $\{A_u' \mid u \in V(G)\}$. A hypergraph G is uniquely determined by its set of subsumed graphs, which provides a convenient visual representation of a hypergraph. In Fig. 2 we have distinguished the u_i and v_i by drawing the v_i as dark dots. We then omitted the u or v, writing only i for u_i or v_i; however, the v_i are in a smaller type to make it easier to distinguish them.

The main result of [2] was the construction of a family of non-reconstructible hypergraphs with $2n + 1$ vertices. In the present notation, the result can be stated as follows.

4.4. Let $n \geqslant 2$ and $m = 0$. Then $X_n = G_n^0(u) + N_n^0(n, 0)$ and $Y_n = G_n^\varepsilon(u) + N_n^1(n, 0)$ are hypomorphic but non-isomorphic.

This is the reason that the definition of $M_j^\varepsilon(n, m)$ and $N_i^\varepsilon(n, m)$ in 2.13 allowed $n = 0$ or $m = 0$. Most of the lemmas of Section 3 dealing with the $M_j^\varepsilon(n, m)$ and $N_i^\varepsilon(n, m)$ do not hold as stated if $m = 0$ or $n = 0$ is to be allowed, but require special cases. Since in this paper we are mainly concerned with $m, n \neq 0$, we do

Table 1.

X	Hypo(X)	$\theta_n^m(X)$	Lemma	Remarks
$G_{j,k}^\varepsilon(n; u)$	$G_{j,k}^{1-\varepsilon}(n; u)$	$G_{j,k}^{1-\varepsilon}(n; u)$	2.5	$j > 1$
$G_{1,k}^\varepsilon(n; u)$	$G_{1,k}^\varepsilon(n; u)$	$G_{1,k}^\varepsilon(n; u)$	2.6	$j = 1$
$M_j^\varepsilon(n, m)$	$M_j^{1-\varepsilon}(n, m)$	$M_j^\varepsilon(n, m)$	3.9	$j < m$
$M_m^\varepsilon(n, m)$	$M_m^\varepsilon(n, m)$	$M_m^{1-\varepsilon}(n, m)$	3,11	$j = m$
$N_i^\varepsilon(n, m)$	$N_i^{1-\varepsilon}(n, m)$	$N_i^\varepsilon(n, m)$	3,10	$i < n$
$N_n^\varepsilon(n, m)$	$N_n^\varepsilon(n, m)$	$N_n^{1-\varepsilon}(n, m)$	3.12	$i = n$

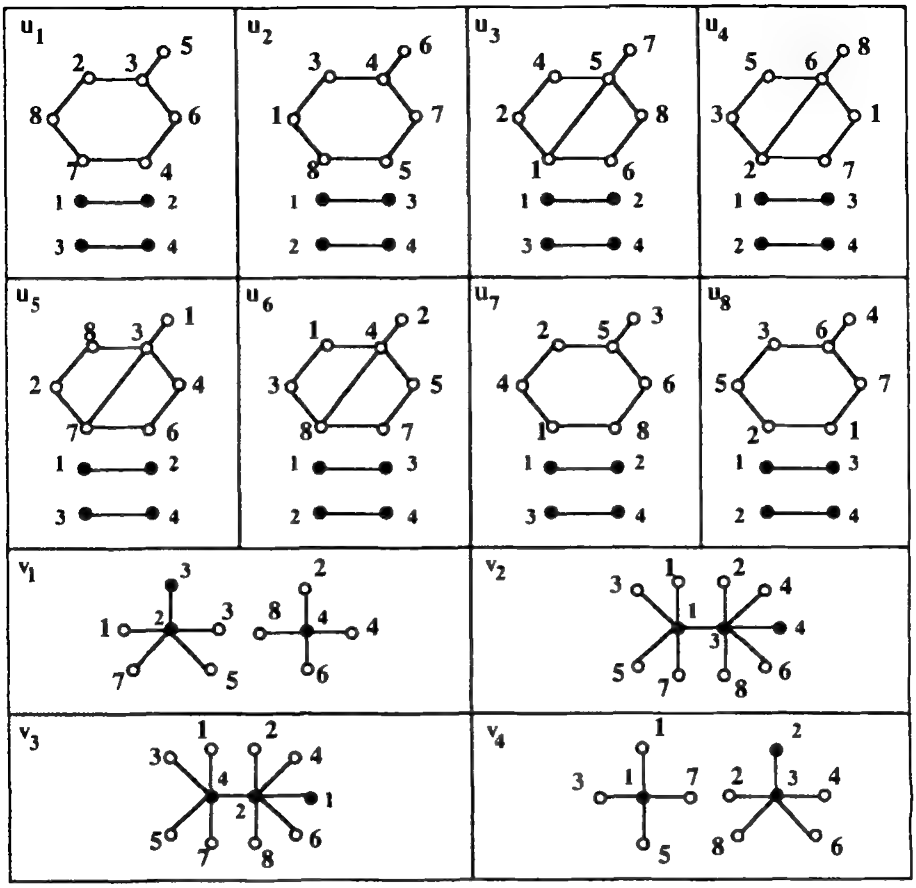


Fig. 2. $X^{(1)}_{2,3,2}$.

not include these special cases, except for this paragraph of explanation, and 4.4 above.

Theorem 4.3 only allows $n \neq m$. If $n = m$, we cannot use the degree argument of 4.3 to force $\psi(V_n(u)) = V_n(u)$ and $\psi(V_m(v)) = V_m(v)$. The easiest way to allow $n = m$ is the following.

Theorem 4.5. *Let $n = m \geqslant 3$ and $1 \leqslant j < n$. Let $W^{\varepsilon}_{j,n} = G^{\varepsilon}_n(u) + G^{\varepsilon}_{2,3}(n;u) + M^{\varepsilon}_j(n,n) + G^{\varepsilon}_n(v)$ and $U^{3}_{j,n} = Hypo(W^{\varepsilon}_{j,n}) = G^{\varepsilon}_n(u) + G^{1-\varepsilon}_{2,3}(n;u) + M^{1-\varepsilon}_j(n,n) + G^{\varepsilon}_n(v)$. Then $W^{\varepsilon}_{j,n} \not\cong U^{\varepsilon}_{j,n}$.*

Proof. By 2.14 and 2.15, $\deg(u_k, G^{\varepsilon}_n(u))$ is either 2^{2n-3} or $2^{2n-3} - 1$, and $\deg(u_k, G^{\varepsilon}_{2,3}(n;u)) = 3$. By 2.13 and 2.16, $\deg(u_k, M^{\varepsilon}_j(n,n)) = deg(v_k, M^{\varepsilon}_j(n,n)) = 2^{n+j-3}$. So $\deg(u_k, W^{\varepsilon}_{j,n}) > \deg(v_k, W^{\varepsilon}_{j,n})$, from which it follows that any putative isomorphism ψ: $W^{\varepsilon}_{j,n} \rightarrow U^{\varepsilon}_{j,n}$ must map $V_n(u)$ to $V_n(u)$ and $V_n(v)$ to $V_n(v)$. By 4.2, ψ must act as θ_n on $V_n(v)$, since only edges of $G^{\varepsilon}_n(v)$ contain only vertices of $V_n(v)$. Since $M^{\varepsilon}_j(n,n)$ is a mixed hypergraph, whereas

$G_n^\varepsilon(u) + G_{2,3}^\varepsilon(n; u)$ is pure, it also follows that $\psi(M_j^\varepsilon(n, n)) = M_j^{1-\varepsilon}(n, n)$ and that $\psi(G_n^\varepsilon(u) + G_{2,3}^\varepsilon(n; u)) = G_n^\varepsilon(u) + G_{2,3}^{1-\varepsilon}(n; u)$. By Table 1, θ_n^n maps $G_n^\varepsilon(u) + G_{2,3}^\varepsilon(n; u)$ to its hypomorph, $G_n^\varepsilon(u) + G_{2,3}^{1-\varepsilon}(n; u)$, so that the set of all such mappings (which includes ψ), is the coset $\theta_n^n \operatorname{Aut}(G_n^\varepsilon(u) + G_{2,3}^\varepsilon(n; u))$. However, ψ cannot equal θ_n^n since $\theta_n^n(M_j^\varepsilon(n, n)) = M_j^\varepsilon(n, n))$, By Table 1.

Now $\operatorname{Aut}(G_n^\varepsilon) = \langle \theta_n \rangle$, by 4.2. Let $\phi \in \operatorname{Aut}(G_n^\varepsilon + G_{2,3}^\varepsilon(n))$. If all such ϕ are also in $\operatorname{Aut}(G_n^\varepsilon)$, then we would need to have $\psi = \theta_n^n$, which is impossible. So if there is an isomorphism ψ, it must act on $V_n(u)$ as $\theta_n\phi$ on V_n, where $\phi \notin \operatorname{Aut}(G_n^\varepsilon)$. Such a ϕ must map some edge of $G_{2,3}^\varepsilon(n)$ to an edge of G_n^ε. By 2.1 and 4.1 it is easy to see that $G_n^\varepsilon = p_{\text{odd}}(G_{n-1}^\varepsilon) + p_{\text{even}}(G_{n-1}^\varepsilon) + G_{n-1,n}^\varepsilon(n)$. Furthermore, $G_{2,3}^\varepsilon(n) = p_{\text{odd}}(G_{2,3}^\varepsilon(n-1)) + p_{\text{even}}(G_{2,3}^\varepsilon(n-1))$, if $n \geq 4$, so that all edges of $G_{2,3}^\varepsilon(n)$ contain only odd or only even vertices, and in $G_n^\varepsilon + G_{2,3}^\varepsilon(n)$, only edges of $G_{n-1,n}^\varepsilon(n)$ contain both odd and even vertices.

Now by 2.10, $\theta_n(A_j^\varepsilon(n)) = A_j^{1-\varepsilon}(n)$. If we compare this with 2.13, keeping in mind that ψ acts as θ_n on $V_n(v)$, we see that ψ must map $p_{\text{odd}}(V_{n-1}(u))$ to $p_{\text{odd}}(V_{n-1}(u))$ and $p_{\text{even}}(V_{n-1}(u))$ to $p_{\text{even}}(V_{n-1}(u))$ if we are to have $\psi(M_j^\varepsilon(n, n)) = M_j^{1-\varepsilon}(n, n)$. Comparing this with $\theta_n\phi$ we see that ϕ must interchange odd and even vertices. It now follows from the previous paragraph that ϕ maps $p_{\text{odd}}(G_{n-1}^\varepsilon + G_{2,3}^\varepsilon(n-1))$ to $p_{\text{even}}(G_{n-1}^\varepsilon + G_{2,3}^\varepsilon(n-1))$, so that we have reduced the action of ϕ to an automorphism of $G_{n-1}^\varepsilon + G_{2,3}^\varepsilon(n-1)$, where $n \geq 4$. To finish the proof (by induction), it is enough to show that $G_3^\varepsilon + G_{2,3}^\varepsilon(3)$ has no automorphisms other than the identity. This is most easily seen from Fig. 3, where $G_3^\varepsilon + G_{2,3}^\varepsilon(3)$ is shown in terms of its subsumed graphs. □

So $W_{j,n}^\varepsilon$ and $U_{j,n}^\varepsilon$ from another family of non-reconstructible hypergraphs. We have only considered m, $n \geq 3$. There are also many pairs of non-reconstructible hypergraphs with m or n equal to 2, not indicated by the above theorems. For example, $G_{1,2}^0(2, u) + M_2^0(2, 2) + M_1^1(2, 2) + N_2^0(2, 2) + G_{1,2}^0(2, v)$ is non-

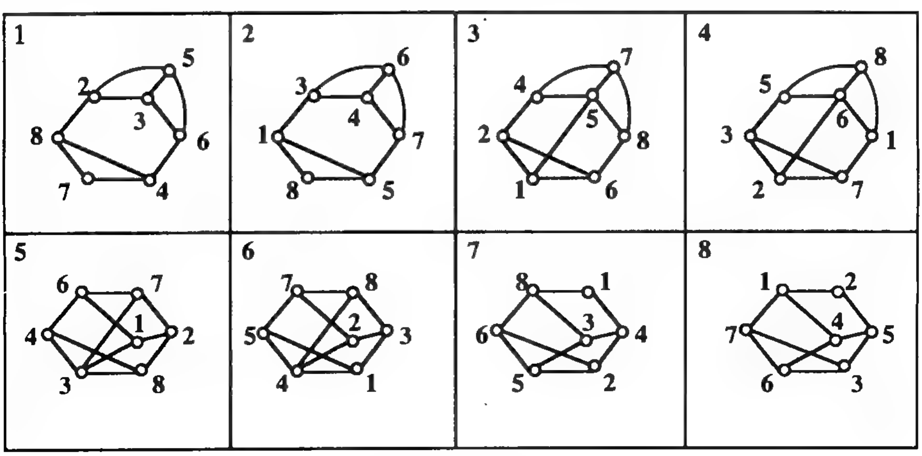


Fig. 3. $G_3^\varepsilon + G_{2,3}^\varepsilon(3)$.

reconstructible, as is $G^1_{1,2}(2, u) + M^1_1(2, 2) + M^1_2(2, 2) + N^1_1(2, 2) + N^0_2(2, 2) + G^0_{1,2}(2, v)$.

Finally, we should like to mention that it would have been virtually impossible to develop these results without the use of a computer. We used the University of Manitoba's Amdahl 5850 to construct the hypergraphs and test for isomorphism. Isomorphism was tested using B.D. McKay's program GLABC of [3].

Acknowledgement

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CONSTRUCTIONS OF SENSITIVE GRAPHS WHICH ARE NOT STRONGLY SENSITIVE

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1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For each a in $V(G)$, let $N(a) = \{x \in V(G) : ax \in E(G)\}$ be the set of *neighbours* of a. A subset S of $V(G)$ is called a *closed set* of G if, for each pair of distinct elements a, b in S, $N(a) \cap N(b) \subseteq S$. Let $\mathscr{L}(G)$ be the family of closed sets of G, inclusive of the empty set $\emptyset$. Evidently, $\mathscr{L}(G)$ is closed under arbitrary intersection and it thus forms a lattice under set-inclusion (see Fig. 1). The lattice $\mathscr{L}(G)$, which was first introduced by Sauer (see [6] and also [3, 4, 5]), is called the *closed-set lattice* of the graph G.

We shall now introduce, in terms of their closed-set lattices, various classes of graphs. A graph G is said to be *minimally critical* if $\mathscr{L}(G) \ncong \mathscr{L}(G-e)$ for each e in $E(G)$, and *maximally critical* if $\mathscr{L}(G) \ncong \mathscr{L}(G+e)$ for any e in $E(\bar{G})$, where $\bar{G}$ is the complement of G. We say that G is *critical* if G is both maximally and minimally critical. A graph G is said to be *sensitive* if for any graph G' that $\mathscr{L}(G) \cong \mathscr{L}(G')$ implies $G \cong G'$. Suppose that G and G' are graphs such that $\mathscr{L}(G) \cong \mathscr{L}(G')$ under a lattice isomorphism Φ. It is easily seen that Φ induces naturally a bijection $\phi : V(G) \to V(G')$ such that for each x in $V(G)$, $\phi(x) = x'$ in $V(G')$ if and only if $\Phi(\{x\}) = \{x'\}$ in $\mathscr{L}(G')$. We call ϕ the *bijection induced* by Φ. A graph G is said to be *strongly sensitive* if for any graph G' and for any lattice isomorphism $\Phi : \mathscr{L}(G) \cong \mathscr{L}(G')$, the bijection ϕ induced by Φ is a graph isomorphism of G onto G'.

For a graph G, the following implications follow immediately from definitions:

$$\begin{aligned} & G \text{ is strongly sensitive} \\ \Rightarrow\; & G \text{ is sensitive} \\ \Rightarrow\; & G \text{ is critical} \\ \Rightarrow\; & G \text{ is } \begin{cases} \text{maximally critical} \\ \text{minimally critical.} \end{cases} \end{aligned}$$

While strongly sensitive graphs are abundant (see [3]), not every graph is maximally critical or minimally critical. There exist maximally critical graphs

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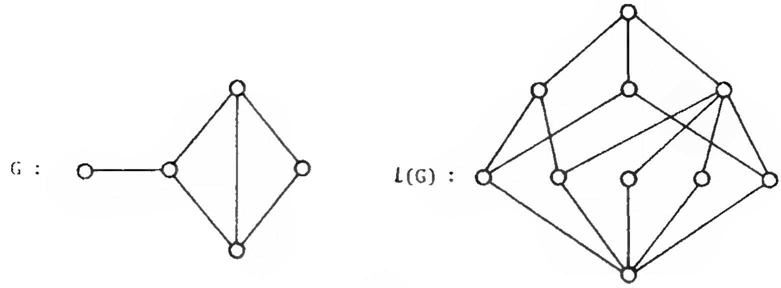

Fig. 1.

which are not minimally critical and vice versa. A class of critical graphs which are not sensitive can also be found in [4]. Now the question is: Are there sensitive graphs which are not strongly sensitive?

It can be verified that the graph of Fig. 1 is the unique graph of minimum order which is sensitive but not strongly sensitive. Two more such graphs of order 6 are shown in Fig. 2. As a matter of fact, it is our main aim in this paper to introduce some methods of construction which produce various families of sensitive graphs that are not strongly sensitive.

Throughout this paper, all graphs are assumed to be finite, nontrivial and simple. For all terminology on graphs and lattices not explained here, we refer to [1] and [2] respectively.

2. Preliminaries

We begin with the following observation:
(*) Let G be a graph of order at least three. Then G is disconnected if and only if $\mathscr{L}(G)$ is direct product decomposable. Thus for graphs G and G' of order at least three such that $\mathscr{L}(G) \cong \mathscr{L}(G')$, G is connected if and only if G' is connected (see

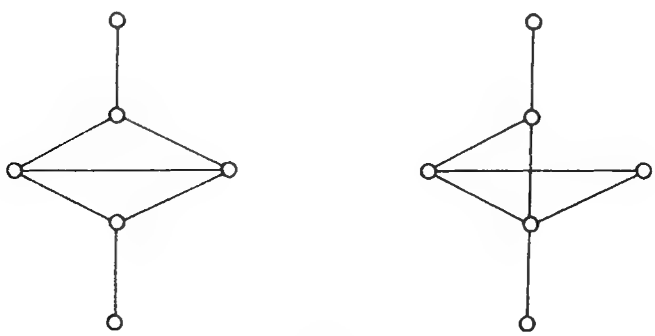

Fig. 2.

[6]). In contrast with this, we have $\mathscr{L}(P_2) \cong \mathscr{L}(O_2)$, where P_2 is the path of order 2 and O_2 is the graph consisting of two isolated vertices.

For a subset X of $V(G)$, the subgraph of G *induced by* X is denoted by $[X]$ and the closed set of G *generated by* X is denoted by $\langle X \rangle$. For graphs G and G', we shall write $\mathscr{L}(G) \overset{(\Phi)}{\cong} \mathscr{L}(G')$ to indicate that lattices $\mathscr{L}(G)$ and $\mathscr{L}(G')$ are isomorphic under the lattice isomorphism Φ. The *image* of X under a mapping ϕ is denoted by $X\phi$.

Lemma 1 [3]. *Let G and G' be graphs such that $\mathscr{L}(G) \overset{(\Phi)}{\cong} \mathscr{L}(G')$ and let $\phi: V(G) \to V(G')$ be the bijection induced by Φ. For any subset X of $V(G)$, any closed set A of G and any vertex a in G, we have:*

(i) $\Phi(\langle X \rangle) = \langle X\phi \rangle$,

(ii) $a \in \langle X \rangle$ *if and only if* $\phi(a) \in \langle X\phi \rangle$,

(iii) $|\langle X \rangle| = |\langle X\phi \rangle|$,

(iv) $\Phi(A) = A\phi$,

(v) $deg_G(a) = 1$ *if and only if* $deg_{G'}(\phi(a)) = 1$, *provided that G is connected and* $|V(G)| \geq 3$,

(vi) $\mathscr{L}([\langle X \rangle]) \overset{(\Phi^*)}{\cong} \mathscr{L}([\langle X\phi \rangle])$, *where Φ^* is Φ confined to* $\mathscr{L}([\langle X \rangle])$.

The following result provides a simpler way to determine whether two closed-set lattices are isomorphic.

Lemma 2 [3]. *Let G and G' be two graphs. Then $\mathscr{L}(G) \cong \mathscr{L}(G')$ if and only if there exists a bijection $\alpha: V(G) \to V(G')$ such that*

$$(\langle \{x, y\} \rangle)\alpha = \langle \{\alpha(x), \alpha(y)\} \rangle$$

for any pair of distinct elements x, y in $V(G)$.

Remark. Given α satisfying the above condition, define $\Phi: \mathscr{L}(G) \to \mathscr{L}(G')$ by $\Phi(S) = S\alpha$. It can be shown that Φ is a lattice isomorphism and the bijection ϕ induced by Φ is identical with α.

3. Rambutans

For any connected graph G of order n, denote by $B(G)$ the graph of order $2n$ obtained by adjoining to each vertex of G a new end vertex. Every graph of the form $B(G)$ is called a *rambutan* (also known as *corona*). It was noted in [4] that every rambutan is critical. It is easy to prove that for any complete graph K_n $(n \geq 2)$, $B(K_n)$ is strongly sensitive. While a class of nonsensitive rambutans was provided in [4], we construct here a class of sensitive rambutans which are not strongly sensitive.

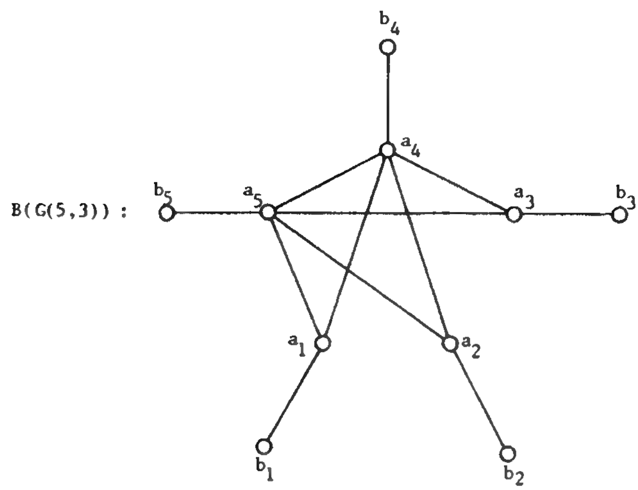

Fig. 3.

Theorem 1. *For each complete graph K_n of order $n \geqslant 4$ with $V(K_n) = \{a_1, a_2, \ldots, a_n\}$ and for each integer r such that $2 \leqslant r \leqslant n-2$, let $G(n, r)$ be a spanning subgraph of K_n with*

$$E(G(n, r)) = E(K_n) - \{a_i a_j : 1 \leqslant i < j \leqslant r\}.$$

Then the rambutan $B(G(n, r))$ is sensitive but not strongly sensitive.

Proof. Let $\{b_1, \ldots, b_n\}$ be the set of end vertices of $B(G(n, r))$ such that $a_i b_i \in E(B(G(n, r)))$ for each $i = 1, \ldots, n$. Let H be any graph such that $\mathscr{L}(B(G(n, r))) \overset{(\Phi)}{\cong} \mathscr{L}(H)$. Let $V(H) = \{x' = \phi(x) : x \in V(B(G(n, r)))\}$. By observation (*), H is connected. By Lemma 1(v), $\{b_1', \ldots, b_n'\}$ forms the set of end vertices of H.

Since $|\langle\{b_i, b_j\}\rangle| = 2$ for all $i, j = 1, \ldots, n$ with $i \neq j$, we have $|\langle\{b_i', b_j'\}\rangle| = 2$. Thus $d(b_i', b_j') \geqslant 3$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Hence $H = B([\{a_1', \ldots, a_n'\}])$.

Claim 1. $a_i' b_i' \in E(H)$ for each $i = r+1, \ldots, n$.

If $a_i' b_i' \notin E(H)$ for some $i = r+1, \ldots, n$, then there exists $j = 1, \ldots, n$ with $j \neq i$ such that $a_j' b_i' \in E(H)$. But then $|\langle\{a_j', b_i'\}\rangle| = 2$ while $|\langle\{a_j, b_i\}\rangle| \geqslant 3$, contradicting Lemma 1(iii).

Claim 2. $a_i' a_j' \in E(H)$ for each $i = r+1, \ldots, n$ and for each $j = 1, \ldots, n$ with $i \neq j$.

Since $|\langle\{b_i, a_j\}\rangle| \geqslant 3$, we have $|\langle\{b_i', a_j'\}\rangle| \geqslant 3$ by Lemma 1(iii). Hence by Claim 1, $a_i' a_j' \in E(H)$.

Claim 3. $a_i'a_j' \notin E(H)$ for all $i, j = 1, \ldots, r$ with $i \neq j$.

Assume on the contrary that $a_i'a_j' \in E(H)$ for some $i, j = 1, \ldots, r$ with $i \neq j$. There exists a $k = 1, \ldots, r$ such that $b_k'a_i' \in E(H)$. But then $|\langle \{b_k', a_j'\} \rangle| \geq 3$ while $|\langle \{b_k, a_j\} \rangle| = 2$, contradicting Lemma 1(iii).

It thus follows that there is a permutation ψ of $\{1, 2, \ldots, r\}$ such that $a_i'a_{\psi(i)}' \in E(H)$ for each $i = 1, \ldots, r$. Note that we have $B(G(n, r)) \cong H$ whatever ψ is, and by Lemma 2, $\mathscr{L}(B(G(n, r))) \overset{(\Phi)}{\cong} \mathscr{L}(H)$. Hence $B(G(n, r))$ is sensitive but not strongly sensitive as ψ may not be the identity mapping. □

4. A construction based on strongly sensitive graphs

There are strongly sensitive graphs in plenty. Indeed, it was proved in [3] that evey C_4-free graph and the covering graph of every lattice are strongly sensitive. In this section, we shall introduce a method to construct sensitive graphs which are not strongly sensitive from a family of strongly sensitive graphs.

Again, let K_n be the complete graph of order $n \geq 4$ with $V(K_n) = \{a_1, a_2, \ldots, a_n\}$. For each integer r such that $2 \leq r \leq n-2$, let $G(n, r)$ be the graph constructed in Section 3. Let $\mathscr{G} = \{G_{r+1}, G_{r+2}, \ldots, G_n\}$ be a given family of $(n-r)$ connected strongly sensitive graphs (thus each G_i is of order at least three). We construct a new graph $G_r^n(\mathscr{G})$ based on $G(n, r)$ and $\mathscr{G}$ as follows:

$$V(G_r^n(\mathscr{G})) = V(G(n, r)) \cup \bigcup_{i=r+1}^{n} V(G_i) \cup \{b_1, b_2, \ldots, b_r\}$$

$G_2^5(G)$:

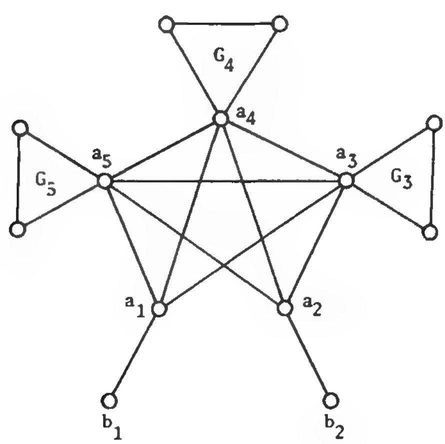

Fig. 4.

where $\{b_1, b_2, \ldots, b_r\}$ is a set of r new vertices and for each $i = r+1, \ldots, n$, a_i is identified with exactly one vertex of G_i and

$$E(G_r^n(\mathscr{G})) = E(G(n, r)) \cup \{a_i b_i : i = 1, 2, \ldots, r\} \cup \bigcup_{i=r+1}^{n} E(G_i).$$

Then we have

Theorem 2. *Every graph $G_r^n(\mathscr{G})$ is sensitive but not strongly sensitive.*

Remark. Graphs of the form $B(G(n, r))$ considered in Theorem 1 are not contained in the family of graphs shown in Theorem 2 since every edge considered as a graph of order 2 is not strongly sensitive.

Proof. Let H be any graph such that $\mathscr{L}(G_r^n(\mathscr{G})) \overset{(\Phi)}{\cong} \mathscr{L}(H)$. Let $V(H) = \{x' = \phi(x) : x \in V(G_r^n(\mathscr{G}))\}$. Again, H must be connected. By Lemma 1(v), $\{b_1', \ldots, b_r'\}$ is a set of end vertices of H. Hence $d(b_i', b_j') \geq 3$ for all $i, j = 1, \ldots, r$ with $i \neq j$. We shall now establish a series of claims:

(1) For each $i = r+1, \ldots, n$ and for all $x, y \in V(G_i)$, $xy \in E(G_i)$ if and only if $x'y' \in E(H)$.

By Lemma 1(vi), $\mathscr{L}(G_i) \overset{(\Phi_i)}{\cong} \mathscr{L}([(V(G_i))\phi])$, where $\Phi_i = \Phi|_{\mathscr{L}(G_i)}$. Since each G_i is strongly sensitive, (1) thus follows.

(2) For all $i = r+1, \ldots, n$, $j = 1, \ldots, r$ and for every $x \in V(G_i)$, $b_j'x' \notin E(H)$.

Assume on the contrary that $b_j'x' \in E(H)$. Since G_i is connected and $|V(G_i)| \geq 3$, there exists y in G_i such that $xy \in E(G_i)$. By (1), $x'y' \in E(H)$. Hence $x', y' \in \langle\{b_j', y'\}\rangle$, and so $x, y \in \langle\{b_j, y\}\rangle$ by Lemma 1(ii). However, this contradicts the fact that $|\langle\{b_j, y\}\rangle \cap V(G_i)| = 1$.

Now, by (2) plus the fact that H is connected and $\{b_1', \ldots, b_r'\}$ is a set of end vertices with $d(b_i', b_j') \geq 3$ for each pair of distinct $i, j = 1, \ldots, r$, we have:

(3) There exists a permutation ψ of $\{1, 2, \ldots, r\}$ such that $a_i'b_{\psi(i)}' \in E(H)$ for each $i = 1, \ldots, r$.

(4) For all $i, j = 1, \ldots, r$ with $i \neq j$, $a_i'a_j' \notin E(H)$.

If $a_i'a_j' \in E(H)$, then $|\langle\{a_i', b_{\psi(j)}'\}\rangle| \geq 3$ but $|\langle\{a_i, b_{\psi(j)}\}\rangle| = 2$, contradicting Lemma 1(iii).

(5) For all $i = r+1, \ldots, n$ and $j = 1, \ldots, r$, $a_i'a_j' \in E(H)$.

Since $|\langle\{a_i, b_{\psi(j)}\}\rangle| \geq 3$, we have $|\langle\{a_i', b_{\psi(j)}'\}\rangle| \geq 3$. By (3) and the fact that $\deg(b_{\psi(j)}') = 1$, $a_i'a_j' \in E(H)$.

(6) For all $i = r+1, \ldots, n$, $j = 1, \ldots, n$ with $i \neq j$ and for every $x \in N(a_i) \cap V(G_i)$, $a_j'x' \notin E(H)$.

Assume on the contrary that $a_j'x' \in E(H)$. By (1), $a_ix \in E(G_i)$ implies that $a_i'x' \in E(H)$. Thus $x' \in \langle\{a_i', a_j'\}\rangle$, contradicting the fact that $x \notin \langle\{a_i, a_j\}\rangle$.

(7) For all $i, j = r+1, \ldots, n$ with $i \neq j$ and for all $x \in V(G_i) - \{a_i\}$ and $y \in V(G_j)$, $x'y' \notin E(H)$.

Suppose $x'y' \in E(H)$. By (6), we need only to consider the following two cases: (i) $x \in V(G_i) - (N(a_i) \cup \{a_i\})$ and $y = a_j$ and (ii) $x \in V(G_i) - \{a_i\}$ and $y \in V(G_j) - \{a_j\}$. In case (i), we have $a_1'a_j' \in E(H)$ by (5). Hence $a_j' \in \langle\{x', a_1'\}\rangle$, contradicting the fact that $a_j \notin \langle\{x, a_1\}\rangle = \{x, a_1\}$. In case (ii), we have $\deg(x) > 1$ since $\deg(x') > 1$. Let $u \in N(x) - \{a_i\}$. By (1), $u'x' \in E(H)$. Hence $x' \in \langle\{u', y'\}\rangle$, contradicting the fact that $x \notin \langle\{u, y\}\rangle = \{u, y\}$.

(8) For all $i, j = r+1, \ldots, n$ with $i \neq j$, $a_i'a_j' \in E(H)$.

Assume on the contrary that $a_i'a_j' \notin E(H)$. Let $x \in N(a_i) \cap V(G_i)$. By (6) and (7), $N(x') \subseteq (V(G_i))\phi$. Again by (7), $N(a_j') \subseteq V(H) - (V(G_i))\phi$. Thus we have $N(x') \cap N(a_j') = \emptyset$. Hence $|\langle\{x', a_j'\}\rangle| = 2$, contradicting the fact that $|\langle\{x, a_j\}\rangle| \geq 3$.

(9) For all $i = r+1, \ldots, n$, $j = 1, \ldots, r$ and for every $x \in V(G_i) - \{a_i\}$, $a_j'x' \notin E(H)$.
If $a_j'x' \in E(H)$, then $a_j' \in \langle\{a_i', x'\}\rangle$, contradicting the fact that $a_j \notin \langle\{a_i, x\}\rangle$.

It thus follows from what we have discussed that $G_r^n(\mathscr{G}) \cong H$ whatever ψ is, without violating the assumption that $\mathscr{L}(G_r^n(\mathscr{G})) \overset{(\Phi)}{\cong} \mathscr{L}(H)$ (see Lemma 2). Since ψ is not necessarily the identity mapping, we conclude that $G_r^n(\mathscr{G})$ is sensitive but not strongly sensitive. $\square$

5. Graphs without cut vertices

All the graphs constructed in the preceding two sections contain end vertices and hence cut vertices. Does there exist a sensitive but not strongly sensitive graph which contains no cut vertices? The answer is 'yes'. Fig. 5 shows the two

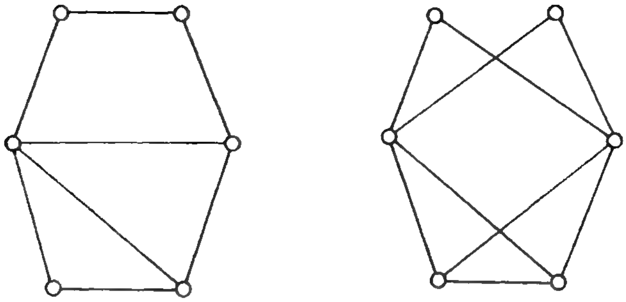

Fig. 5.

such graphs of minimum order. In this section, we shall furnish a class of such graphs. For each vertex x in G, we shall denote by $N[x]$ the *closed neighbourhood* of x, i.e. $N[x] = N(x) \cup \{x\}$.

Theorem 3. *Let G be a graph as shown in Fig. 6 where*
(i) *each G_i $(i = 1, 2, 3)$ is a strongly sensitive graph,*
(ii) *$a_1 \in V(G_1) - N[a]$, $b_1 \in V(G_2) - N[b]$ and $c_1 \in V(G_3) - N[c]$.*
Then G is sensitive but not strongly sensitive.

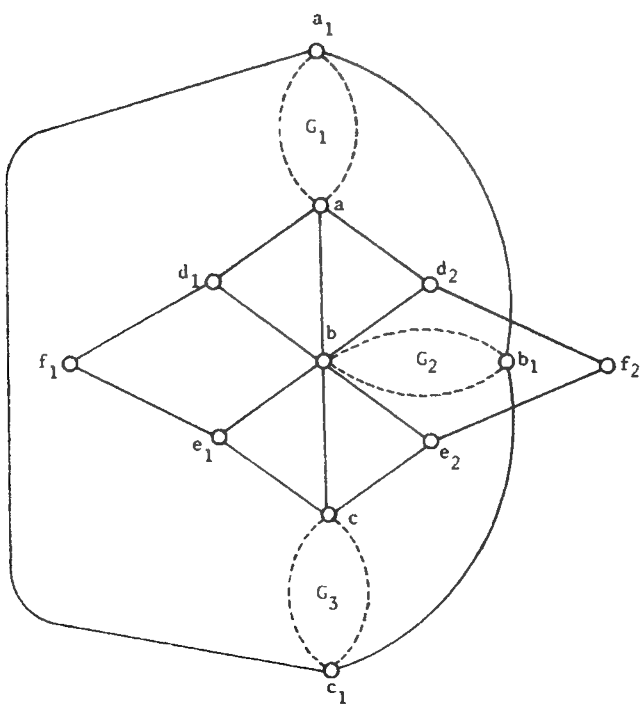

Fig. 6.

Proof. Let H be any graph such that $\mathscr{L}(G) \overset{(\Phi)}{\cong} \mathscr{L}(H)$. Let $V(H) = \{x' : x \in V(G)\}$. Clearly, H is connected. Again, we shall establish a series of claims.

(1) For each $i = 1, 2, 3$ and for all $x, y \in V(G_i)$, $xy \in E(G)$ if and only if $x'y' \in E(H)$.

This follows immediately from Lemma 1(vi) and the fact that each G_i is strongly sensitive.

(2) For all $i = 1, 2$, $j = 1, 2, 3$ and for every $x \in V(G_j)$, $f_i'x' \notin E(H)$.

It suffices to consider the case when $i = j = 1$. Assume on the contrary that $f_1'x' \in E(H)$. Since G_1 is connected and $|V(G_1)| \geq 3$, there exists $y \in V(G_1) \cap N(x)$. By (1), $x'y' \in E(H)$. Thus $x', y' \in \langle\{f_1', y'\}\rangle$, contradicting the fact that $|\langle\{f_1, y\}\rangle \cap V(G_1)| = 1$.

(3) $N(f_1') = \{d_i', e_k'\}$ and $N(f_2') = \{d_j', e_r'\}$ where $\{i, j\} = \{k, r\} = \{1, 2\}$.

Since $\deg(f_i) \neq 1$ for each $i = 1, 2$, we have $\deg(f_i') \neq 1$ by Lemma 1(v). Hence $\deg(f_i') \geq 2$ for each $i = 1, 2$. Since $\langle\{f_1, f_2\}\rangle = \{f_1, f_2\}$, we have $\langle\{f_1', f_2'\}\rangle = \{f_1', f_2'\}$ and hence $N(f_1') \cap N(f_2') = \emptyset$. Moreover, $f_1'f_2' \notin E(H)$, for otherwise by letting $u' \in N(f_1') - \{f_2'\}$, we have $u' \in \{d_1', d_2', e_1', e_2'\}$ by (2), and hence $|\langle\{u', f_2'\}\rangle| \geq 3$, contradicting the fact that $|\langle\{u, f_2\}\rangle| = 2$. Thus $N(f_1') \cup N(f_2') = \{d_1', d_2', e_1', e_2'\}$ and $\deg(f_1') = \deg(f_2') = 2$. Since $\langle\{d_1, d_2\}\rangle = \{d_1, d_2, a, b\}$, we have $\langle\{d_1', d_2'\}\rangle = \{d_1', d_2', a', b'\}$ and hence we conclude that: (i) $d_1'f_1'$ and $d_2'f_1'$ cannot be both in $E(H)$ and (ii) $d_1'f_2'$ and $d_2'f_2'$ cannot be both in $E(H)$. Similarly, by considering $\langle\{e_1, e_2\}\rangle = \{e_1, e_2, b, c\}$, we have: (iii) $e_1'f_1'$ and $e_2'f_1'$ cannot be both in $E(H)$ and (iv) $e_1'f_2'$ and $e_2'f_2'$ cannot be both in $E(H)$. Thus (3) is established.

(4) For every pair of distinct vertices x, y in $\{d_1, d_2, e_1, e_2\}$, $x'y' \notin E(H)$.

Since $|\langle\{x, f_i\}\rangle| = |\langle\{y, f_i\}\rangle| = 2$ for each $i = 1, 2$, we have $|\langle\{x', f_i'\}\rangle| = |\langle\{y', f_i'\}\rangle| = 2$ for each $i = 1, 2$. Now (4) follows from (3).

(5) For each $i = 1, 2$, $a'e_i', c'd_i' \notin E(H)$.

Assume on the contrary that $a'e_i' \in E(H)$ for some $i = 1, 2$. Let $x \in V(G_1) \cap N(a)$. Then $ax \in E(H)$ implies that $a'x' \in E(H)$ by (1). Thus $|\langle\{x', e_i'\}\rangle| \geq 3$, contradicting the fact that $|\langle\{x, e_i\}\rangle| = 2$. Hence $a'e_i' \notin E(H)$. By symmetry, we have $c'd_i' \notin E(H)$ for each $i = 1, 2$.

(6) $a'd_i', b'd_i', b'e_i', c'e_i', a'b', b'c' \in E(H)$ for each $i = 1, 2$.

M_4 :

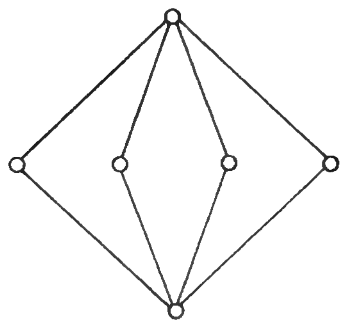

Fig. 7.

Since $\mathscr{L}([\{a, b, d_1, d_2\}]) \cong \mathscr{L}([\{b, c, e_1, e_2\}]) \cong M_4$ (see Fig. 7), we have $\mathscr{L}([\{a', b', d_1', d_2'\}]) \cong \mathscr{L}([\{b', c', e_1', e_2'\}]) \cong M_4$. As $d_1'd_2', e_1'e_2' \notin E(H)$ by (4), (6) thus follows.

(7) $a_1'b_1', a_1'c_1', b_1'c_1' \in E(H)$.

Observe that $[\langle\{a_1, b_1, c_1\}\rangle] = [\{a_1, b_1, c_1\}] \cong C_3$, which is a strongly sensitive graph. By Lemma 1(vi), we have $[\langle\{a_1', b_1', c_1'\}\rangle] = [\{a_1', b_1', c_1'\}] \cong C_3$, establishing (7).

(8) For each $i = 1, 2$ and for every $x \in (V(G_1) - \{a\}) \cup (V(G_2) - \{b\})$, $d_i'x' \notin E(H)$.

This follows immediately from the fact that $(V(G_i))\phi$ is a closed set of H for each $i = 1, 2$.

Similarly, we have

(9) For each $i = 1, 2$ and for every $x \in (V(G_2) - \{b\}) \cup (V(G_3) - \{c\})$, $e_i'x' \notin E(H)$.

(10) For each $i = 1, 2$ and for every $x \in V(G_3) - \{c\}$, $d_i'x' \notin E(H)$.

Since $\langle\{a, x\}\rangle = \{a, x\}$, we have $\langle\{a', x'\}\rangle = \{a', x'\}$ and hence $d_i'x' \notin E(H)$.

Similarly, we have

(11) For each $i = 1, 2$ and for every $x \in V(G_1) - \{a\}$, $e_i'x' \notin E(H)$.

(12) $a'b_1', a'c_1', b'a_1', b'c_1', c'a_1', c'b_1' \notin E(H)$.

This follows from the fact that $(V(G_1))\phi$, $(V(G_2))\phi$ and $(V(G_3))\phi$ are closed sets of H.

(13) $a'c' \notin E(H)$.

If $a'c' \in E(H)$, then by letting $x \in V(G_1) \cap N(a)$, we have $a'x' \in E(H)$ and hence $a' \in \langle \{c', x'\} \rangle$, contradicting the fact that $a \notin \langle \{c, x\} \rangle = \{c, x\}$.

(14) For all $x \in V(G_2) - \{b\}$ and $y \in V(G_2) - \{b_1\}$, $a'x'$, $a_1'y'$, $c'x'$, $c_1'y' \notin E(H)$.

Since $(V(G_2))\phi$ is a closed set of H, the above claim is true.

(15) For all $x \in V(G_3) - \{c\}$ and $y \in V(G_3) - \{c_1\}$, $a'x'$, $a_1'y' \notin E(H)$.

Suppose $a'x' \in E(H)$. Then $\deg(x') > 1$ in H, which implies that $\deg(x) > 1$ in G by Lemma 1(v). Let $u \in N(x) - \{c\}$. Then $u'x' \in E(H)$ and hence $x' \in \langle \{a', u'\} \rangle$, contradicting the fact that $x \notin \langle \{a, u\} \rangle = \{a, u\}$. Since $(V(G_3))\phi$ is a closed set of H, we have $a_1'y' \notin E(H)$.

Similarly, we have

(16) For all $x \in V(G_1) - \{a\}$ and $y \in V(G_1) - \{a_1\}$, $c'x'$, $c_1'y' \notin E(H)$.

(17) For all $x \in V(G_1) - \{a, a_1\}$, $y \in V(G_2) - \{b, b_1\}$ and $z \in V(G_3) - \{c, c_1\}$, $x'y'$, $y'z'$, $x'z' \notin E(H)$.

Suppose $x'y' \in E(H)$. Then $\deg(x') > 1$, which implies that $\deg(x) > 1$. Let $u \in N(x) - \{a\}$. Then $u'x' \in E(H)$ by (1). Hence $x' \in \langle \{u', y'\} \rangle$, contradicting the fact that $x \notin \langle \{u, y\} \rangle$. Similarly, $y'z'$, $x'z' \notin E(H)$.

We thus conclude from the previous discussion that $G \cong H$. Observe that there are in (3) four different combinations for $N(f_1')$ and $N(f_2')$, which show that the bijection ϕ induced by Φ need not be the identity mapping. Moreover, by Lemma 2, the assumption that $\mathscr{L}(G) \overset{(\Phi)}{\cong} \mathscr{L}(H)$ is not violated when each of these four combinations holds for $N(f_1')$ and $N(f_2')$. Therefore G is sensitive but not strongly sensitive. □

Remark. Condition (ii) in Theorem 3 cannot be replaced by the following:

(*) $\quad a_1 \in V(G_1) - \{a\}, \quad b_1 \in V(G_2) - \{b\},$ and
$\quad c_1 \in V(G_3) - \{c\}$.

For instance, the graph G of Fig. 8, which satisfies condition (i) in Theorem 3 and condition (*) above, is not even maximally critical ($\mathscr{L}(G) \cong \mathscr{L}(G + e)$, where $e = xy$).

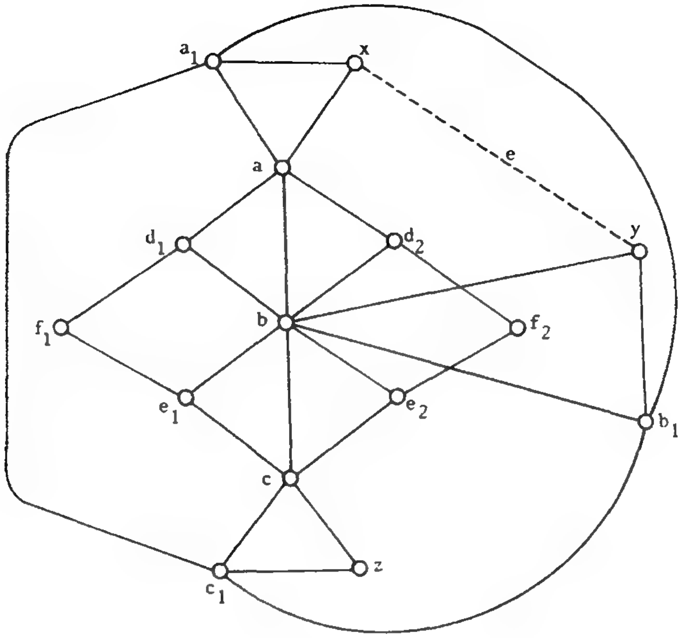

Fig. 8.

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COMBINATORIAL RESOLUTION OF SYSTEMS OF DIFFERENTIAL EQUATIONS. IV. SEPARATION OF VARIABLES

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In the context of the combinatorial theory of ordinary differential equations recently introduced by the authors, a concrete interpretation is given to the classical method of separation of variables. This approach is then extended to more general equations and applied to systems of differential equations with forcing terms.

1. Introduction

Classically, the ordinary differential equation

$$y' = dy/dt = f(t)g(y), \quad y(0) = \alpha \tag{1.1}$$

is said to have 'separable variables' since we can write, in an informal manner, (see [4], pp. 12–20)

$$dy/g(y) = f(t)\,dt \tag{1.2}$$

and integrate on both sides to get

$$\Phi(y) = \int_\alpha^y du/g(u) = \int_0^t f(x)\,dx \tag{1.3}$$

If we now let $\Phi^{\langle -1\rangle}$ denote the inverse function of Φ, the solution of (1) can be simply written as

$$y(t) = \Phi^{\langle -1\rangle}\left(\int_0^t f(x)\,dx\right). \tag{1.4}$$

In this paper we establish the validity of this solution by giving a direct combinatorial interpretation of the function $y = \Phi^{\langle -1\rangle}(t)$ as the solution of the

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autonomous differential equation

$$y' = g(y), \qquad y(0) = \alpha \tag{1.5}$$

and then by showing that the combinatorial solution of (1.1) admits a kind of 'separation of vertices' which implies (1.4). Moreover, we show that this combinatorial separation of variables can be applied to more general differential equations, or systems of differential equations, of the form

$$y = \sum_{J} f_j(t) g_j(y) \tag{1.6}$$

where j varies over some finite index set J. Examples include the linear equation

$$y' = f(t)y + g(t) \tag{1.7}$$

and equations of control theory such as

$$y' = g_0(y) + u(t)g_1(y) \tag{1.8}$$

where g_0 and g_1 are given and $u(t)$ is a variable function.

In particular we establish by purely combinatorial methods a functional expansion for the solution of (1.6), in terms of iterated integrals, which implies the so-called fundamental formula of Fliess for systems of differential equations with forcing terms ([1, 2]).

As in our previous papers [8, 9], we work in a combinatorial setting where functions are replaced by species of structure over linearily ordered sets ($\mathbb{L}$-species). The relationship with analysis is obtained by taking generating functions of $\mathbb{L}$-species, a process which preserves all the usual operations of calculus, including integration, when properly defined on $\mathbb{L}$-species. The reader is referred to [8], where the combinatorial theory of differential equations has been initiated and which serves as a basis for the present paper, for background and terminology.

2. Combinatorial separation of variables

We first consider the example of the differential equation

$$Y' = T \cdot Y^3, \qquad Y(0) = 1 \tag{2.1}$$

whose analytical solution is given by

$$y(t) = 1/\sqrt{1-t^2}. \tag{2.2}$$

The use of capital letters in (2.1) indicates the fact that we want a combinatorial solution, in terms of $\mathbb{L}$-species. As shown in [8], there is a unique $\mathbb{L}$-species $Y = A_{TY^3}(T)$ which satisfies (2.1): the equation can be rewritten in the integral form

$$Y = 1 + \int_0^T XY^3(X)\, dX \tag{2.3}$$

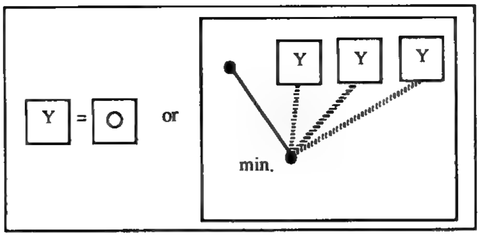

Fig. 1.

and interpreted as follows (see Fig. 1), using the standard combinatorial interpretation of the integral (see [8, §2] and Fig. 5]:

- there is a unique A_{TY^3}-structure on the empty set, denoted by ○;
- any non-empty A_{TY^3}-structure can be canonically identified with the composite structure consisting of
  - (a) the minimum element of the underlying set
  - (b) a singleton point
  - (c) an ordered triple (left, middle, right) of similar A_{TY^3}-structures.

One can then attach the singleton point and the ordered triple of A_{TY^3}-structures to the minimum element (then called a *fertile point*) as shown in Fig. 1. Iterating this procedure we find that A_{TY^3}-structures essentially consist of certain *increasing labelled rooted planar trees* (with labels increasing from root to leaves) such as the one illustrated by Fig. 2, on the linearly ordered set $[19] = \{1, 2, \ldots, 19\}$.

Let $Y = \mathrm{Ter}(T)$ denote the $\mathbb{L}$-species of *increasing ternary trees,* that is the solution of the autonomous equation

$$Y' = Y^3, \qquad Y(0) = 1. \tag{2.4}$$

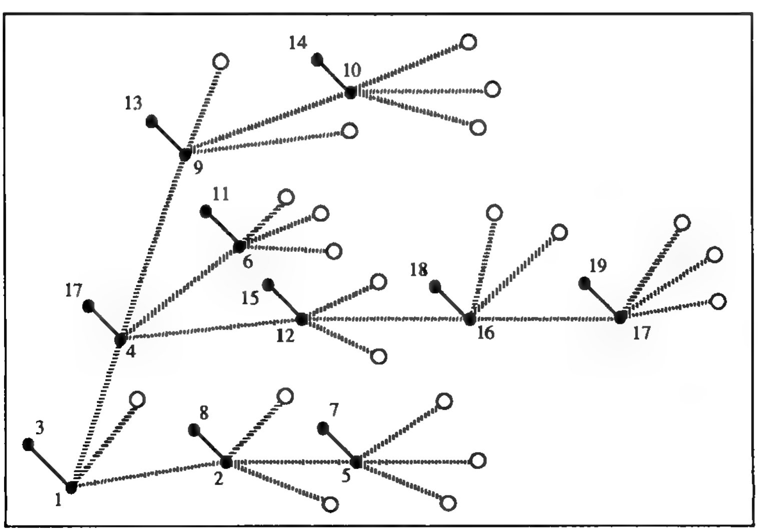

Fig. 2.

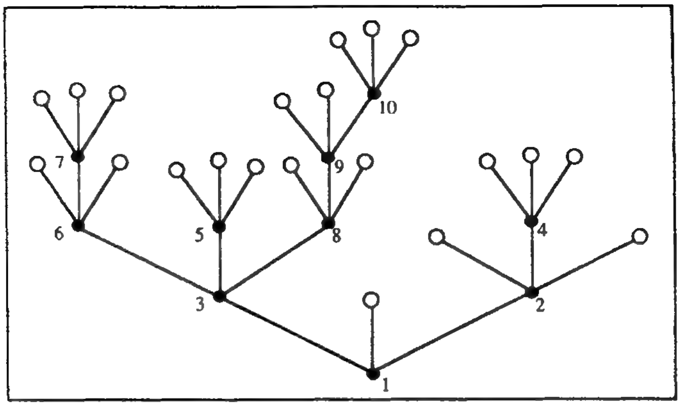

Fig. 3.

They are similar to A_{TY^3}-structures but simpler since there are no singletons attached to the roots; an example is given in Fig. 3.

For $n \geqslant 0$, let t_n be the number of increasing ternary trees with n internal vertices. It is easily seen that $t_n = (2n-1)t_{n-1}$, for $n \geqslant 1$, and that $t_0 = 1$. Thus

$$t_n = (2n-1) \cdots (3)(1) \tag{2.5}$$

and we obtain the generating function,

$$\begin{aligned} \mathrm{Ter}(t) &= \sum_{n \geqslant 0} t_n t^n / n! \\ &= \frac{1}{\sqrt{1-2t}} \end{aligned} \tag{2.6}$$

by the binomial theorem (see also [8], example 5.5).

A comparison of Figs 2, 3 and 4 shows that an A_{TY^3}-structure can be viewed as an incrasing ternary tree where vertices have been replaced by cells containing

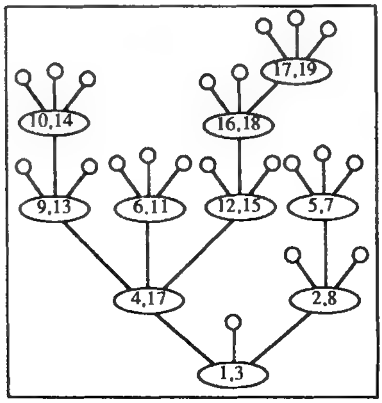

Fig. 4.

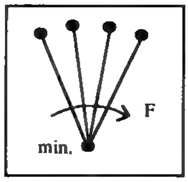

Fig. 5.

(unordered) pairs of elements, these cells being linearly ordered according to their minimum elements. This corresponds to substitution of species: we have

$$A_{TY^3}(T) = \mathrm{Ter}\left(\frac{T^2}{2!}\right) \tag{2.7}$$

where, as usual, equality stands for isomorphism of species.

Using (2.6), we conclude that the generating function for A_{TY^3} is

$$\begin{aligned} A_{TY^3}(t) &= \mathrm{Ter}\left(\frac{t^2}{2}\right) \\ &= \frac{1}{\sqrt{1-t^2}} \end{aligned} \tag{2.8}$$

as expected, and also, by the binomial theorem, that

$$A_{TY^3}(t) = \sum_{n\geq 0} \frac{[(2n-1)\cdots(3)(1)]^2 t^{2n}}{2n!} \tag{2.9}$$

Note that $\frac{1}{2}T^2 = \int_0^T X\,dX$ and that, more generally, a structure of species $\int F = \int_0^T F(X)\,dX$ can be represented as in Fig. 5.

We end this section with the basic classical result on the method of separation of variables which is surprisingly simple from a combinatorial point of view. It comes as a natural generalization of the previous example and of example 4.5.a of [8] (the homogeneous linear equation).

Proposition 2.1. *Let $Y = A_{G(Y)}(T)$ be the solution of the autonomous differential equation*

$$Y' = G(Y), \qquad Y(0) = Z; \tag{2.10}$$

then the solution $Y = A_{F(T)G(Y)}(T)$ of the equation with separable variables

$$Y' = F(T)G(Y), \qquad Y(0) = Z \tag{2.11}$$

is given by

$$A_{F(T)G(Y)}(T) = A_{G(Y)}\left(\int_0^T F(X)\,dX\right). \tag{2.12}$$

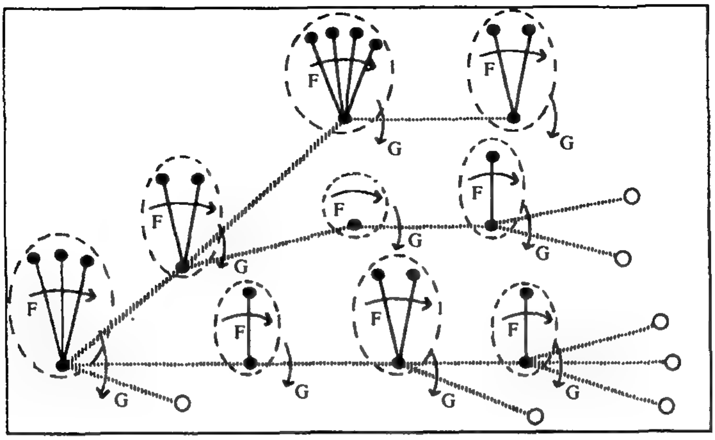

Fig. 6.

Proof. Simply observe that the increasing $F(T)G(Y)$-enriched arborescences that occur as the canonical solution of (2.11) can be viewed as $A_{G(Y)}(\int F)$-structures. See Fig. 6. $\square$

3. Iterated integrals

Iterated integrals such as

$$\int_0^T F(S)\int_0^S G(R)\int_0^R H(X)\,dX\,dR\,dS \tag{3.1}$$

occur in the expansion of product integrals (see [3]) and in the solution of certain systems of differential equations (see e.g. [5], chap. 14). The standard combinatorial interpretation of the integral can be applied to interpret iterated integrals (3.1) as $\mathbb{L}$-species. One gets enriched increasing arborescences as in Fig. 7.

More generally, given a family $F=\{F_j\}_{j\in J}$ of $\mathbb{L}$-species, where J is some finite set, and given any word $\alpha\in J^*$, there is an induced iterated integral, denoted by

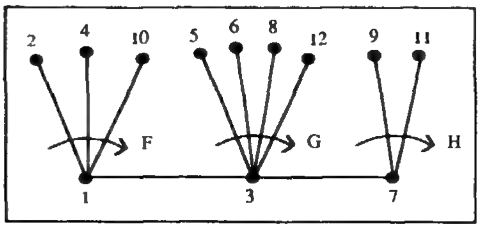

Fig. 7.

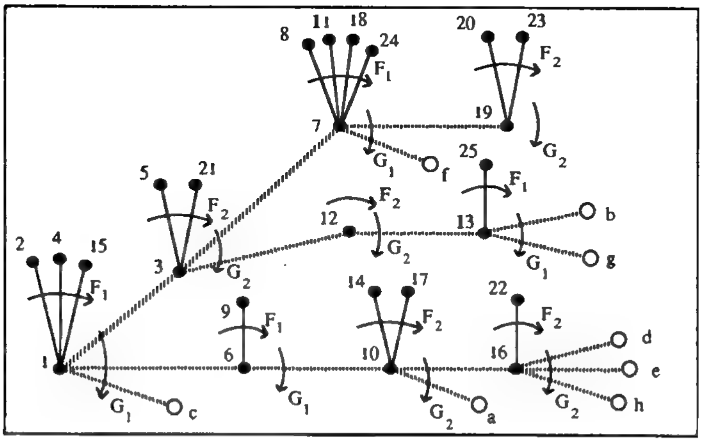

Fig. 8.

$\int_0^T F\,d\alpha$, which is defined inductively as follows:

If $\alpha = \varepsilon$, the empty word, then

$$\int_0^T F\,d\varepsilon = 1 \tag{3.2}$$

the 'empty set' species;

If $\alpha = j\beta$, i.e. $j \in J$ is the first letter of the word α, then

$$\int_0^T F\,d(j\beta) = \int_0^T F_j(X) \int_0^X F\,d\beta\,dX. \tag{3.3}$$

An example of an $\int_0^T F\,d\alpha$-structure is given in Fig. 9 (see below), with $\alpha = 1\ 2\ 1\ 1\ 2\ 2\ 1\ 2\ 2$. It is readily seen that an $\int_0^T F\,d\alpha$-structure is an assembly of $(\sum_{j\in J} \int_0^T F_j(X)\,dX)$-structures aligned in the natural order of their smallest elements. Hence we can state:

Proposition 3.1. *For any finite family $\{F_j\}_{j\in J}$ of $\mathbb{L}$-species, we have*

$$\sum_{\alpha\in J^*} \int_0^T F\,d\alpha = \exp\left(\sum_{j\in J} \int_0^T F_j(X)\,dX\right). \tag{3.4}$$

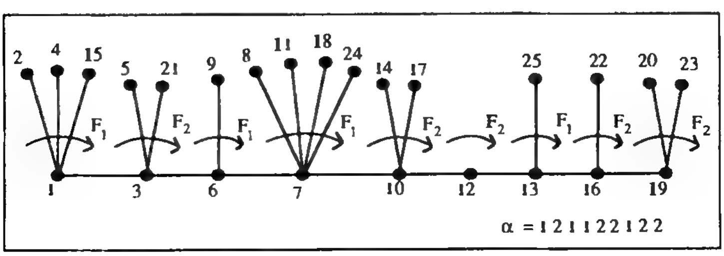

Fig. 9. $I(T)$-structure.

4. A generalized combinatorial separation of variables

The technique of combinatorial separation of variables can also be applied to equations, or systems of equations, of the form

$$Y' = \sum_j F_j(T)G_j(Y), \qquad Y(0) = Z \tag{4.1}$$

where j varies over some finite index set J. We will first deal with one simple equation:

$$Y' = F_1(T)G_1(Y) + F_2(T)G_2(Y), \qquad Y(0) = Z \tag{4.2}$$

with $J = \{1, 2\}$, and later give appropriate generalizations as well as examples, in Section 5.

Setting $M = M(T, Y) = F_1(T)G_1(Y) + F_2(T)G_2(Y)$, the canonical solution of (4.2) is the $\mathbb{L}$-species $Y = A_M = A_M(T, Z)$ of so-called *M-enriched increasing arborescences with buds* (buds constitute an extra sort of elements, corresponding to the variable Z; see §3 of [8]). A typical A_M-structure is represented by Fig. 8 where it is seen that the M-enrichment at each fertile point consists of either an $F_1(T)G_1(Y)$- or an $F_2(T)G_2(Y)$-structure.

Now combinatorial separation of variables can be applied as follows: associate with each M-enriched increasing arborescence a pair of structures as illustrated by Figs 9 and 10, for the arborescence of Fig. 8, thus defining two species $I(T)$ and $H(Z)$.

The $I(T)$-structure is in fact an iterated integral $\int_0^T F\,d\alpha$-structure obtained by extracting the $\int F_1$- and $\int F_2$-structures of the arborescence and aligning them in their natural order. Moreover, we take note of this relative order in what remains of the arborescence to obtain a structure on buds (see Fig. 10). This is the $H(Z)$-structure.

It can be seen that the $H(Z)$-structure comes from the application to Z of a sequence of operators of the type $\mathcal{D}_1$ or $\mathcal{D}_2$, according to the same word α as in the iterated integral; the $\mathcal{D}_j$'s are 'eclosion operators' (terminology due to G. Labelle; see [6, 7] and [8, §5]) defined, for $j \in J$, by $\mathcal{D}_j = G_j(Z)\,\partial/\partial Z$ (Fig. 11).

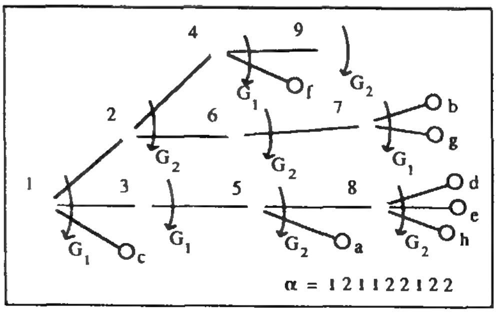

Fig. 10. $H(Z)$-structure.

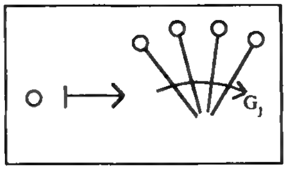

Fig. 11.

For instance the $H(Z)$-structure of Fig. 10 is a typical structure of the species

$$(\mathscr{D}_2 \circ \mathscr{D}_2 \circ \mathscr{D}_1 \circ \mathscr{D}_2 \circ \mathscr{D}_2 \circ \mathscr{D}_1 \circ \mathscr{D}_1 \circ \mathscr{D}_2 \circ \mathscr{D}_1)(Z). \tag{4.3}$$

Note that we compose operators from right to left so that the order of α is actually reversed. We introduce the notation $\mathscr{D}_\alpha$ for the operators defined inductively by

$$\mathscr{D}_\varepsilon = Id, \qquad \mathscr{D}_{(j\beta)} = \mathscr{D}_\beta \circ \mathscr{D}_j. \tag{4.4}$$

To summarize, combinatorial separation of variables associates to any M-enriched increasing arborescence, a word α and a pair of structures of species $\int_0^T F\,d\alpha$ and $\mathscr{D}_\alpha(Z)$ respectively. This correspondence is clearly reversible and bijective and also valid for an index set J of any finite cardinality. Hence we have:

Proposition 4.1. *Let $Y = A_M(T, Z)$ be the solution of the differential equation*

$$Y' = \sum_{j\in J} F_j(T)G_j(Y), \qquad Y(0) = Z \tag{4.5}$$

where $\{F_j\}_{j\in J}$ and $\{G_j\}_{j\in J}$ are given finite families of $\mathbb{L}$-species. Then we have

$$A_M(T, Z) = \sum_{\alpha\in J^*} \left(\int_0^T F\,d\alpha\right)(\mathscr{D}_\alpha(Z)) \tag{4.6}$$

and, more generally, for any $\mathbb{L}$-species Φ,

$$\Phi(A_M(T, Z)) = \sum_{\alpha\in J^*} \left(\int_0^T F\,d\alpha\right)(\mathscr{D}_\alpha(\Phi(Z)). \tag{4.7}$$

Proof. Only (4.7) remains to be proved. For any word α, the expression $\int_0^T F\,d\alpha\, \mathscr{D}_\alpha$ can be interpreted as an operator on $\mathbb{L}$-species; it should be clear that when this operator is applied to any $\Phi(Z)$-structure, the result is a $\Phi(A_M(T, Z))$-structure. This induces the desired isomorphism. □

Remark. We now show that formula (4.7) is also valid in the case of a system of differential equations of the form (4.5) where

- Y is a vector of species $Y = (Y_1, \ldots, Y_n)$
- each F_j is an $\mathbb{L}$-species as before,
- each G_j is a vector $(G_{j,1}, \ldots, G_{j,n})$ of species of n sorts
- Z is a vector $(Z_1, \ldots, Z_n)$ of variables that correspond to n sorts of buds.

The system (4.5) can then be written as

$$Y_1' = \sum_{j \in J} F_j(T) G_{j,i}(Y_1, \ldots, Y_n), \qquad Y_i(0) = Z_i, \quad i = 1, \ldots, n \tag{4.8}$$

As observed in [8, §6], the combinatorial solution of a system of equations is similar to that of a single equation. It consists, in the present case, of enriched increasing arborescences as in Fig. 8, with the following additional features:

A ‘color’, i.e. an element of the set $\{1, \ldots, n\}$, is attributed to each fertile point and the structure attached to a fertile point of color i is required to be a $F_i(T)G_{j,i}(Y_1, \ldots, Y_n)$-structure for some $j \in J$. The ith component $Y_i = A_{M,i}(T, Z)$ of the solution $Y = A_M(T, Z)$ consistes of structures which are either a single bud of sort i or such colored arborescences whose root has color i.

For each $j \in J$, the ‘eclosion’ operator $\mathcal{D}_j$ is defined by

$$\mathcal{D}_j = \sum_{i=1}^{n} \frac{G_{j,i}(Z_1, \ldots, Z_n)\partial}{\partial Z_i} \tag{4.9}$$

it applies to species $\Phi(Z) = \Phi(Z_1, \ldots, Z_n)$ of n sorts of buds as follows: a $\mathcal{D}_j(\Phi(Z))$-structure is a $\Phi(Z)$-structure in which, for some color i, a bud of sort i has been replaced by a $G_{j,i}(Z_1, \ldots, Z_n)$-structure.

With this interpretation, formula (4.7) is then again valid. This constitute a combinatorial proof of the ‘fundamental formula’ of Michel Fliess (see [1, 2]) for systems of differential equations with forcing terms, that is systems of the form (4.8), with $J = \{0, 1, \ldots, m\}$, $F_0(T) = 1$ and, for $j \geq 1$, $F_j(T) = U_j(T)$ is a variable ‘input’ function.

5. Examples

The linear differential equation

$$Y' = F(T)Y + G(T), \qquad Y(0) = Z \tag{5.1}$$

was studied from a combinatorial point of view in [8] and the classical formulas proven. However this is a special case of (4.5), with $J = \{1, 2\}$, $F_1(T) = F(T)$, $F_2(T) = G(T)$, $G_1(Y) = Y$, and $G_2(Y) = 1$, and the application of (4.6) gives an alternate formula for the solution of (5.1) which is closer to the simple combinatorial solution. Indeed, in this case, we have $\mathcal{D}_1 = Z\partial/\partial Z$ and $\mathcal{D}_2 = \partial/\partial Z$ and the only words $\alpha \in J^*$ for which the operator $\mathcal{D}_\alpha$ applied to Z yields a non zero result are

$$1\ 1 \cdots 1 = 1^k \quad \text{and} \quad 1\ 1 \cdots 1\ 2 = 1^k 2, \quad k \geq 0. \tag{5.2}$$

For these words, we have

$$\mathcal{D}_{(1^k)}(Z) = Z \quad \text{and} \quad \mathcal{D}_{(1^k 2)}(Z) = 1. \tag{5.3}$$

Hence we get the following expression (to be compared with formula (4.16) of [8]) for the solution of (5.1).

Proposition 5.1. *The solution $Y = A_{F,G}(T, Z)$ of the linear differential equation* (5.1) *can be expressed as*

$$A_{F,G}(T, Z) = \sum_{k\geq 0} \int_0^T (F, G)\, d(1^k)Z + \sum_{k\geq 0} \int_0^T (F, G)\, d(1^k 2) \tag{5.4}$$

where, for $\alpha \in \{1, 2\}^*$, $\int_0^T (F, G)\, d\alpha$ denotes the iterated integral associated with α and the family $(F_1, F_2) = (F, G)$.

We now give an application to differential equation with forced entry of the form

$$Y' = G(Y) + U(T), \qquad Y(0) = 0 \tag{5.5}$$

and in particular to the equation

$$Y' = aY + bY^2 + U(T), \qquad Y(0) = 0 \tag{5.6}$$

that models an electric circuit with a quadratic resistance. $U(T)$ represents a variable 'entry' or 'control' function (the current, in (5.6)), and $Y(T)$, the output (voltage) function. These equations are of the form (4.5), with $J = \{0, 1\}$, $F_0(T) = 1$, $F_1(T) = U(T)$, $G_0(Y) = G(Y)$ and $G_1(Y) = 1$; thus $\mathcal{D}_0 = G(Z)\, \partial/\partial Z$ and $\mathcal{D}_1 = \partial/\partial Z$. By proposition 4.1, the solution $Y = V(T)$ of the differential equation (5.5) can be expressed as

$$V(T) = \sum_{\alpha\in\{0,1\}^*} \int_0^T (1, U)\, d\alpha\, \mathcal{D}_\alpha(Z)\, |_{Z=0}. \tag{5.7}$$

The evaluation at $Z = 0$ in (5.7) comes from the initial condition $V(0) = 0$.

We see that $V(T)$ depends not only on T but also on the variable function $U(T)$, i.e. $V = V(T, U)$. Thus (5.7) is a functional expansion of $V(T, U)$ in terms of iterated integrals $\int_0^T (1, U)\, d\alpha$, for words $\alpha \in \{0, 1\}^*$ and the family $(F_0, F_1) = (1, U)$. For the associated generating series, we get an analogous functional expansion

$$V(t, U) = \sum_{\alpha\in\{0,1\}^*} V_\alpha \int_0^t (1, U)\, d\alpha \tag{5.8}$$

which we call the *Fliess series* of the system 5.5. Much attention is devoted in control theory to the numerical computation of the coefficients V_α in the Fliess series. In particular, Fliess and his school (see [1, 2]) transform (5.5) into an algebraic equation satisfied by the formal power generating series of $\{V_\alpha\}_{\alpha\in J^*}$

$$S = \sum_{\alpha\in\{0,1\}^*} V_\alpha X_\alpha \tag{5.9}$$

in non commuting variables X_0 and X_1 (the monomials X_α are defined recursively

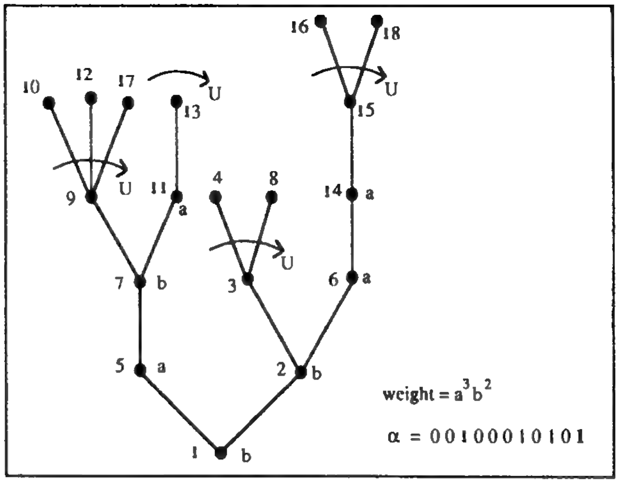

Fig. 12.

by $X_\varepsilon = 1$ and $X_{j\beta} = X_j X_\beta$). The equation is of the form $S = f(S)$ and can be solved iteratively as a fixed point problem using a programmed computer.

The approach proposed here can greatly simplify the computation (see [2]) of the coefficients V_α since it provides a direct combinatorial interpretation for them. Consider, for example, the equatoin (5.6), where a and b are formal parameters. The combinatorial solution $Y = V(T)$ consists of increasing arborescences with the following properties (see Fig. 12):

• No buds appear, since, by the initial condition, $V(0) = 0$.

• There are two types (0 or 1) of fertile points. Fertile points of type 0 have 1 or 2 sons (left, right) and carry a weight, a or b, accordingly; these sons are themselves fertile. Those of type 1 are simply the roots of $\int U$-structures; they carry no weight.

• The global weight of the structure is the product of the weights of the fertile points of type 0.

The word $\alpha \in \{0, 1\}^*$ is obtained by reading successively the types of the fertile points in their natural order. Combinatorial separation of variables then gives, on the one hand, a $\int_0^T (1, U)\, d\alpha$-structure (Fig. 13) and, on the other hand, a

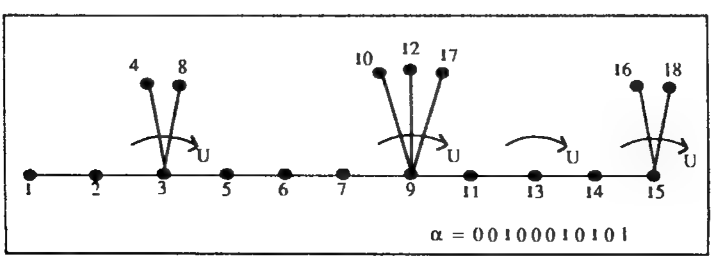

Fig. 13.

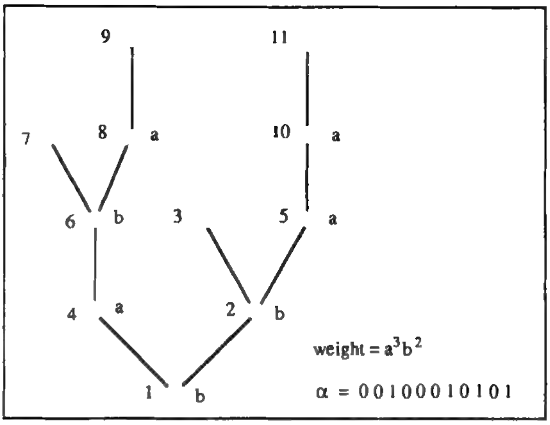

Fig. 14.

so-called 'weighted increasing 1–2 planar tree' on $k = |\alpha|$ phantom buds (Fig. 14). The latter has labels corresponding to the order of enclosures, which can be of two kinds:

$$\mathscr{D}_0 = \frac{(aZ + bZ^2)\partial}{\partial Z} \quad \text{and} \quad \mathscr{D}_1 = \frac{\partial}{\partial Z}. \tag{5.10}$$

The word α and the weight $a^i b^j$ are the same as those of the arborescence and can be read on the associated tree.

For $\alpha \in \{0, 1\}^*$, let $P(\alpha)$ be the set of all such 1-2 trees over α and let $P_\alpha(a, b)$ be its generating polynomial:

$$P_\alpha(a, b) = \sum_{p \in P(\alpha)} \text{weight}(p). \tag{5.11}$$

As formula (5.7) shows, $P(\alpha)$ is the set of empty (i.e. no buds) $\mathscr{D}_\alpha(Z)$-structures, that is

$$P(\alpha) = \mathscr{D}_\alpha(Z)|_{Z=0}. \tag{5.12}$$

In terms of generating functions, we obtain the following:

Corollary 5.2. *The coefficients V_α in the Fliess series* (5.8) *for the solution $Y = V(T)$ of the equation $Y' = aY + bY^2 + U(T)$, $Y(0) = 0$ are the generating polynomials of* 1–2 *trees over α*:

$$V_\alpha = P_\alpha(a, b). \tag{5.13}$$

In conclusion, we note that the polynomials $P_\alpha(a, b)$ can be given a more explicit form using standard combinatorial techniques such as Motzkin words and paths and Dyck words (see [10]).

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LABELING ANGLES OF PLANAR GRAPHS

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A well-known theorem of Heawood states that 3-edge-coloring bridgeless planar cubic graphs—and, hence, the four-color theorem—is equivalent to labeling vertices with either +1 or −1 so that the sum around any face is 0 (mod 3). In this paper we introduce the notion of "angle-labeling" and give results analogous to Heawood's for bridgeless planar graphs with vertices of degree 2 or 3.

1. Introduction

The four-color theorem has had a long and fruitful history. The fecundity of the four-color problem has been splendidly documented in the award-winning paper "Thirteen Colorful Variations on Guthrie's Four-Color Conjecture" by Saaty which appeared in 1972 [4], and which was the basis for a later book [5].

Two of the variations which are of interest to us in the present paper are the edge-colorings of P.G. Tait and the vertex-labelings of P.J. Heawood. Tait initiated the study of edge-coloring in 1880 by translating the problem of coloring maps to the problem of coloring the edges of cubic graphs [6]. In so doing he mistakenly believed that he had proved the four-color theorem. The following theorem is proved in [1, pp. 26–27].

Theorem 1 (Tait). *The four-color theorem is equivalent to the statement that every bridgeless planar cubic graph is 3-edge-colorable.*

Since the four-color theorem is now known to be true, it follows that every bridgeless planar cubic graph is 3-edge-colorable. We offer the following conjecture which extends this result.

Conjecture. Let G be a bridgeless planar graph each of whose vertices is of degree 2 or 3 and having at least two vertices of degree 2. Then G is 3-edge-colorable.

(We note that such a graph having exactly one vertex of degree 2 is not

3-edge-colorable, for the same reason that a cubic graph that has a bridge is not 3-edge-colorable.)

The main goal of this paper is to present a method of attack on the problem of proving this conjecture. Our approach is to mimic the way in which Heawood translated the problem of coloring the edges of a graph to the problem of solving a system of congruences arising from the labeling of the vertices [2].

Theorem 2 (Heawood). *A bridgeless planar cubic graph is* 3*-edge-colorable if and only if its vertices can be labeled either* 1 *or* 2 *in such a way that the sum of the labels around any face is divisible by* 3.

2. Angle-labeling of cubic graphs

We begin our discussion of angle-labeling of cubic graphs by stating a result from [3] in which an "angle-labeling" of a graph means that the "angles" of the graph are labeled either 0 or 1. (Since an angle can be defined formally as a pair of adjacent edges, we need not require the graph to be planar.) With each vertex of the graph we associate six angles, three "interior" angles and three "exterior" angles, as shown in Fig. 1. We abuse notation slightly and allow a_i to represent both the angle and its label.

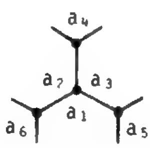

Fig. 1.

Theorem 3 (Loupekine and Watkins). *A cubic graph is* 3*-edge-colorable if and only if there is an angle-labeling of the graph with* 0's *and* 1's *such that at each vertex the six associated angle-labels satisfy the congruence*

$$a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \equiv 2 \pmod 4.$$

We now state a result for planar cubic graphs in which we label each angle of the graph either 1 or 2 and, with each edge of the graph, we associated four angles, as shown in Fig. 2. This result follows immediately from Theorem 6 in the next section.

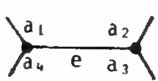

Fig. 2.

Theorem 4. *A bridgeless planar cubic graph G is* 3*-edge-colorable if and only if there is an angle-labeling of the graph with* 1's *and* 2's *such that*

(a) *at each edge the four associated angle-labels satisfy*

$$a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod 2;$$

(b) *the sum of the angle-labels interior to any face of G is* 0 (mod 3).

3. Angle-labeling of graphs having vertices of degree 2 and 3

We now turn to graphs that have vertices of degree 2 as well as vertices of degree 3.

Theorem 5. *Let G be a bridgeless planar graph each of whose vertices is of degree 2 or 3. Then, G is 3-edge-colorable if and only if there is an angle-labeling of the graph with* 1's *and* 2's *such that*

(a) *the sum of the angle-labels interior to any face is* 0 (mod 3);

(b) *the sum of the angle-labels around any vertex is* 0 (mod 3).

Proof. First we assume that the angles of G have been labeled with 1's and 2's so that conditions (a) and (b) hold. We color the edges of G with colors 0, 1 and 2 as follows. Begin by coloring some edge with color 0. Then, for an edge e_2 adjacent to some previously colored edge e_1, we color e_2 with the sum (mod 3) of the color of e_1 and the label (or labels) of the angle (or angles) to the right of the path e_1e_2. We then repreat this process, always coloring a new edge that is adjacent to an edge that has been colored previously.

We show that this procedure leads to a 3-edge-coloring of G. Let v be the vertex incident to each of e_1 and e_2. If v has degree 2 then, by condition (b), one angle at v has label 1 and the other angle at v has label 2; thus, whichever label is to the right in going from e_1 to e_2, we get a color for e_2 that differs from the color for e_1. If v has degree 3, then by condition (b) all the angles at v have the same label; thus, whether there is one angle or two angles to the right in going from e_1 to e_2, we get a color for e_2 that differs from the color for e_1. Furthermore, it follows from condition (a) that the color assigned to an edge is independent of the path used to reach that edge. Thus, G is 3-edge-colorable.

In order to prove the converse, we assume that the edges of G have been 3-colored with colors 0, 1 and 2. Then to label the angle between two edges—say e_1 and e_2, colored respectively c_1 and c_2, with e_2 counterclockwise from e_1—we label the angle with $c_2 - c_1 \pmod 3$. This labeling certainly satisfies condition (a) since, if the edges clockwise around any face are colored $c_1, c_2, \ldots, c_k$, then the sum of the angles around this face is $(c_2 - c_1) + (c_3 - c_2) + \cdots + (c_1 - c_k) = 0$. This labeling also satisfies condition (b): at a vertex of degree 2, the two adjacent edges are colored c_1 and c_2, say, and one angle is labeled $c_2 - c_1$ and the other $c_1 - c_2$, so one label is 1 and the other label is 2; at a vertex of degree 3, the three edges are colored, in counterclockwise order, either 0, 1, 2 or 0, 2, 1; in the first

case the three angles are all labeled 1, and in the second case the three angles are all labeled 2. This completes the proof of the theorem. □

In the preceding argument we always gave the same label to all three angles around a vertex of degree 3; this leads to the following restatement of Theorem 5.

Corollary. *Let G be a bridgeless planar graph each of whose vertices is of degree* 2 *or* 3. *Then, G is* 3*-edge-colorable if and only if there is a labeling with* 1's *and* 2's *of the vertices of degree* 3 *and of the angles at vertices of degree* 2 *in which one angle is labeled* 1 *and the other angle is labeled* 2, *such that*

> *around any face the sum of the vertex-labels (for vertices of degree* 3) *and the interior angle-labels (for vertices of degree* 2) *is* 0(mod 3).

We can now give the generalization of Theorem 4 for graphs having vertices of degree 2 or 3. This characterization is in terms of the four angles associated with an edge (see Fig. 2).

Theorem 6. *Let G be a bridgeless planar graph each of whose vertices is of degree* 2 *or* 3. *Then, G is* 3*-edge-colorable if and only if there is an angle-labeling of the graph with* 1's *and* 2's *such that*

(a) *the sum of the angle-lables interior to any face is* 0 (mod 3);
(b) *for each edge e joining two vertices v and w the four angle-labels associated with e satisfy*

$$a_1 + a_2 + a_3 + a_4 \equiv \deg(v) + \deg(w) \pmod 2.$$

Proof. First we assume that G is 3-edge-colorable. Then, by Theorem 5, there is an angle-labeling such that condition (a) holds and also the sum of the angle-labels around any vertex is 0 (mod 3). If a vertex has degree 3, therefore, all three angles have the same label; whereas, if a vertex has degree 2, then one angle is labeled 1 and the other 2. Thus, if both v and w have degree 3, then the sum of the four angles at an edge is 0 (mod 2); if, say, v has degree 3 and w has degree 2, then the angle sum is odd; if both v and w have degree 2, then the angle sum is 6. So, condition (b) holds as well.

Conversely, suppose that the angles of G have been labeled so that conditions (a) and (b) hold. We show that G has a 3-edge-coloring. For each edge e, let v be the vertex at one end of e and let a_1 and a_4 be the associated angle-lables at that end. Then we label e with $a_1 + a_4 + \deg(v) + 1 \pmod 2$. (Because of condition (b), we get the same label for e if we use the other end and label e with $a_2 + a_3 + \deg(w) + 1 \pmod 2$.) This labeling of the edges partitions the edges of G into two classes, those labeled 0 and those labeled 1. We claim that the edges labeled 1 form a disjoint set of cycles.

Let v be a vertex of degree 3 and let a_1, a_2, a_3 be the three angles at v. Then the sum of the labels for the three edges incident to v is $2a_1 + 2a_2 + 2a_3 + 3\deg(v) + 3 \equiv 0 \pmod 2$. Thus, at a vertex of degree 3, either all three edges are labeled 0 or two edges are labeled 1 and one edge is labeled 0. Now let v be a vertex of degree 2, and let a_1 and a_2 be the two angles at v. Then the sum of the labels for the two edges incident to v is $2a_1 + 2a_2 + 2\deg(v) + 2 \equiv 0 \pmod 2$. Thus, at a vertex of degree 2, both edges have the same label. It follows that the edges labeled 1 form a disjoint set of cycles in G.

Thus, by the Jordan Curve theorem, we can color the regions between these cycles black or white so that these cycles form the boundary between regions colored black and regions colored white. If an angle is in a white region, then we do not change its label. If an angle is in a black region, then replace its label a by $3-a$. Condition (a) still holds. Because of the way in which edges were labeled, an edge e incident to a vertex of degree 3 is labeled 1 if $a_1 \neq a_4$, and is labeled 0 if $a_1 = a_4$. Therefore, if all three edges incident to such a vertex are labeled 0, then all three angles have the same label, and, since such a vertex is an interior point to a white or black region, this remains true after our color switch. On the other hand, if two of the three edges are labeled 1, then the label of the angle between these edges differs from the label of the other two angles. However, after the color switch, all three have the same label. Thus, we can assign a label 1 or 2 to each vertex of degree 3—namely, the label of the surrounding angles.

Now, condition (a) is simply the condition given in the corollary to Theorem 5, so by that corollary, G is 3-edge-colorable. □

We conclude with a result that combines angle and edge labeling. As labels, we use elements of the symmetric group

$$S_3 = \{a, b : a^3 = b^2 = 1,\ ab = ba^2\},$$

in which 1 is the identity element. We label angles with either a or a^2 and we label the "sides" of edges with either 1 or b. Thus, we associate with each edge four angle-labels and two edge-labels; with each face, we associate an angle-label at each vertex on the boundary and an edge-label for each edge on the boundary.

Theorem 7. *Let G be a bridgeless planar graph each of whose vertices is of degree 2 or 3. Then, G is 3-edge-colorable if and only if there is a labeling of G with elements from S_3 in which the angles of G are labeled a and a^2 and the sides of edges are labeled* 1 *and b such that*

(a) *the product in S_3 about any face of all the interior angle and edge labels (taken in order) is the identity* 1;

(b) *for each edge e the total number of a's and b's is congruent to the sum* $\pmod 2$ *of the degrees of the endpoints of e. (Note that a^2 counts as two a's.)*

Proof. First we suppose that G can be 3-edge-colored. Label both sides of each edge with the identity 1. Then label the angles with 1's and 2's so that the two conditions of Theorem 6 hold. If at each angle we now replace its label x with a^x, then conditions (a) and (b) are immediately satisfied.

Conversely, assume that G has been labeled so that condition (a) and (b) hold. The idea is to reverse the above procedure, but first any b's that occur as labels have to be eliminated. By condition (a) the number of b's around any face is even. We can eliminate the b's one pair at a time by replacing the label b with the identity 1 and 'sliding' the b counter-clockwise around the interior of the boundary of the face, changing any angle-label we pass from a to a^2 (or from a^2 to a), until the b reaches another edge labeled b, at which point we replace the b^2 with the identity 1. Since $ab = ba^2$ (and $a^2b = ba$), this maintains condition (a) at each stage and, since an edge losing or gaining one b also loses or gains an a, the total number of a's and b's does not change (mod 2). In this way we remove all b's. Now, at each angle, replace the label a^x with x. Conditions (a) and (b) of Theorem 6 now hold, so G can be 3-edge-colored. □

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MAXIMAL INDUCED TREES IN SPARSE RANDOM GRAPHS

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A study of the orders of maximal induced trees in a random graph G_p with small edge probability p is given. In particular, it is shown that the giant component of almost every G_p, where $p = c/n$ and $c > 1$ is a constant, contains only very small maximal trees (that are of a specific type) and very large maximal trees. The presented results provide an elementary proof of a conjecture from [3] that was confirmed recently in [4] and [5].

1. Introduction

Denote by $\mathscr{G}(n, p)$ the set of all graphs with vertex set $V = \{1, 2, \ldots, n\}$ in which the edges are chosen independently and with probability p. In other words, if G is a graph with vertex set V and it has m edges, then

$$P(\{G\}) = p^m(1-p)^{\binom{n}{2}-m}.$$

Let G_p stand for a random graph from $\mathscr{G}(n, p)$. We say that almost every (a.e.) graph in $\mathscr{G}(n, p)$ has a certain property Q if $P(Q) \to 1$ as $n \to \infty$.

What is the probability distribution for the orders of induced subgraphs in a.e. graph G_p? A lot of interesting results devoted to this fundamental question in random graph theory one can find in [9] where the case p-constant was examined in great detail.

In this paper we shall be assuming that the edge probability p depends on n and $p(n) \to 0$ as $n \to \infty$. Under this assumption we focus our attention on the orders of maximal induced trees that are contained in a.e. graph G_p. A tree that is not properly contained in any larger tree is called a maximal tree. Since we deal here only with induced trees therefore the word 'induced' will be very often omitted.

The symbols o and O are with respect to n unless stated otherwise. All logarithms are natural. Also $\lfloor x \rfloor$ means an integer part of x. Finally, throughout the paper $\gamma(n)$ stands for a sequence tending to infinity (usually arbitrarily slowly) as $n \to \infty$.

2. Maximal induced trees

Given a random graph G_p, what numbers are likely and are not likely to occur as orders of maximal trees in G_p? This section is devoted to an examination of this question.

We say that a graph has property $\mathcal{T}_k$ if it contains a maximal tree of order k. Let us begin our considerations with the following result.

Theorem 1 ([3]). *Assume that* $0<p<1$ *and* $\varepsilon>0$ *are fixed. Then a.e. graph* G_p *has property* $\mathcal{T}_k$ *for every integer* k *satisfying*

$$(1+\varepsilon)\frac{\log n}{\log d}<k<(2-\varepsilon)\frac{\log n}{\log d}$$

where $d=1/q$.

Notice also that a.e. G_p is such that for every set S of $s<(1-\varepsilon)\dfrac{\log n}{\log d}$ vertices there is a vertex $v \notin S$ joined to precisely one vertex in S. Thus, in particular, every tree of order $s<(1-\varepsilon)\dfrac{\log n}{\log d}$ is contained in a tree of order $s+1$. Furthermore (see e.g. [3]) almost no G_p contains a tree on more than $(2+\varepsilon)$ $\dfrac{\log n}{\log d}$ vertices.

The above situation becomes more involved in a case when $p=p(n)\to 0$ as $n\to\infty$. Assume first that $0<p\leqslant c/n$ where $0<c\leqslant 1$ is a constant. Then a random graph G_p is very sparse and contains many components which are trees. In fact, suppose $p=c/n$, $0<c<1$, and $\gamma(n)\to\infty$. Then a.e. G_p is such that every component is a tree or a unicyclic graph and there are at most $\gamma(n)$ vertices on the unicyclic components (see e.g. [2, p. 99]). Obviously every isolated tree is itself a maximal tree and every unicyclic component contains at least three maximal trees. A wide variety of results devoted to possible orders of isolated trees in such a random graph the reader can find in Bollobás [2, Ch. V. 3]. As an example let us quote ([2, p. 106 and 108]) the following fact. (We say that a graph has property $\mathcal{J}_k$ if it contains a tree component on k vertices).

Theorem 2. *Assume that* $p=1/n$.

(i) *If* $k_0/n^{2/3}\to\infty$, *then almost no* G_p *has* $\mathcal{J}_k$ *for every* $k\geqslant k_0$.
(ii) *If* $k_0/n^{2/3}\to 0$, *then a.e.* G_p *has* $\mathcal{J}_k$ *for some* $k\geqslant k_0$.
(iii) If $k_0/n^{2/7}\to 0$, *then a.e.* G_p *has* $\mathcal{J}_k$ *for every* $k\leqslant k_0$.

Now let us turn our attention to a case when a.e. graph G_p is the union of one giant component, small unicyclic components and small tree components. Namely, we shall be assuming that $p=c/n$ where $c>1$ is a constant. Denote by $L_i(G_p)$ the order of the ith largest component of G_p. Then (see e.g. [2, p. 138]) we know that the order of the giant component of a.e. G_p satisfies

$$|L_1(G_p)-(1-t(c))n|\leqslant\gamma(n)n^{\frac{1}{2}}$$

where $\gamma(n) \to \infty$ and

$$t(c) = \frac{1}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k. \tag{1}$$

Moreover, for every fixed $i \geq 2$

$$\left| L_i(G_p) - \frac{1}{a} (\log n - \tfrac{5}{2} \log \log n) \right| \leq \gamma(n) \tag{2}$$

where

$$a = c - 1 - \log c. \tag{3}$$

Our first result specifies those numbers that are not likely to occur as orders of maximal trees in the giant component. Let

$$h(c, \alpha) = (1 - \alpha)^{1 - 1/\alpha} c \exp\left[-\frac{c\alpha}{2} - (1 - \alpha) c e^{-c\alpha} \right]. \tag{4}$$

It can be checked that, for a given c, there is $\alpha_0 = \alpha_0(c)$ such that the function $h(c, \alpha)$ is increasing for $0 < \alpha < \alpha_0$ and is decreasing for $\alpha_0 < \alpha < 1$. Moreover $h(c, \alpha_0) > 1$. Let $\alpha(c)$ be smaller positive root of the equation

$$h(c, \alpha) = 1, \quad 0 < \alpha < 1. \tag{5}$$

The following result is true.

Theorem 3. *Let $p = c/n$ where $c > 1$ is a constant and $\gamma(n) \to \infty$. Then the giant component of a.e. graph G_p contains no maximal trees of order k, where*

$$\gamma(n) \leq k \leq \alpha(c) n. \tag{6}$$

Proof. Let X_k be the number of all trees on k vertices in G_p. Then the expected value of X_k is given by

$$E(X_k) = \binom{n}{k} k^{k-2} p^{k-1} (1 - p)^{\binom{k}{2} - (k-1)}. \tag{7}$$

Furthermore, let Y_k stand for the number of those trees of order k which are maximal in G_p and, in addition, for every such tree there is a vertex in G_p connected with at least two of its vertices. It is clear that every maximal tree contained in the giant component is of this kind. Moreover

$$E(Y_k) \leq E(X_k)(n - k) \binom{k}{2} p^2 [1 - kp(1 - p)^{k-1}]^{n-k-1}.$$

In order to get a good upper bound for $E(Y_k)$ we use the following consequence of Stirling's formula: if $\gamma(n)$ is a sequence tending to infinity as $n \to \infty$, then for all large n and all α such that αn is integer and $\gamma(n) \leq \alpha n \leq n - \gamma(n)$ it is the case

that

$$\binom{n}{\alpha n} \leq \frac{1}{\sqrt{2n\alpha(1-\alpha)}} (\alpha^{\alpha}(1-\alpha)^{1-\alpha})^{-n}. \tag{8}$$

Consequently, making the change of variable $k = \alpha n$ and applying the inequality $1 - x \leq e^{-x}$ together with the formula

$$1 - x = \exp\left[-x - \frac{x^2}{2} + O(x^3)\right] \tag{9}$$

we find that, for all large n,

$$E(Y_k) = O\left\{\frac{1}{\sqrt{\alpha n}} [h(c, \alpha)]^{\alpha n}\right\}$$

where $h(c, \alpha)$ is given by (4). Now, for all α such that $\gamma(n)/n \leq \alpha < \alpha(c)$ we have $h(c, \alpha) < 1$ and consequently the summation of $E(Y_k)$ over all k satisfying (6) tends to zero as $n \to \infty$. This implies the thesis of the theorem. □

Remark. The above proof gives also some additional information about the orders of small components of G_p. Namely, for $p = c/n$, $c > 1$, the unicyclic components that appear in a.e. G_p are of fixed orders only.

The last theorem suggests that the giant component of a random graph $G_{c/n}(c > 1)$ can contain only some maximal trees of small orders and some very large maximal trees. It appears that the small maximal trees that can be contained in the giant component are of a very special structure. We introduce the following definition. Let T be a tree in a graph G and $N(T)$ be the set of all neighbors of T. Assume that $|N(T)| = 1$ and, in addition, that the vertex from $N(T)$, which is called here a *horn*, is connected with exactly two vertices from T. Then $T \cup N(T)$ will be called a *horn-tree*. We have the following theorem.

Theorem 4. *Let $p = c/n$ where $c > 1$ is a constant. Then the only maximal trees of order less than $\gamma(n)$ which can appear in the giant component of G_p are horn-trees.*

Proof. The probability that G_p contains a subset on k vertices, where $4 \leq k \leq \gamma(n)$ and $\gamma(n) \to \infty$ arbitrarily slowly as $n \to \infty$, that spans more than k edges is at most

$$\sum_{k=4}^{\gamma(n)} \binom{n}{k} \sum_{l=k+1}^{\binom{k}{2}} \binom{\binom{k}{2}}{l} p^l (1-p)^{\binom{k}{2}-l}$$

$$\leq \sum_{k=4}^{\gamma(n)} \left(\frac{en}{k}\right)^k \binom{k}{2} \binom{\binom{k}{2}}{k+1} p^{k+1}$$

$$\leq \frac{1}{n} \sum_{k=4}^{\gamma(n)} \left(\frac{e^2 c}{2}\right)^k k^3 c = o(1).$$

Clearly, this implies our thesis. □

Observe that every horn-tree contained in the giant component of G_p is a cycle $v_1v_2 \cdots v_iv_1$ where v_j, $1 \leq j \leq i-1$, is the root of an isolated tree in $G_p \setminus \{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_i\}$ and v_i is the horn which belongs to the giant component. In other words these 'trees' hang on to the giant component by their horns. This, together with the last two theorems, implies easily the following result.

Theorem 5. *Assume $p = c/n$, where $c > 1$ is a constant. Then the giant component of a.e. graph G_p contains a tree on at least $\lfloor \alpha(c)n \rfloor + 1$ vertices where $\alpha(c)$ is defined by* (5).

Proof. Having in mind Theorems 3 and 4 it suffices to show that the number of all horn-trees contained in the giant component is finite. In fact, if this is the case, then picking up an edge that is not in a horn-tree (a lot of such edges will be available) one can surely extend this edge to a tree on at least $\lfloor \alpha(c)n \rfloor + 1$ vertices. First we determine the expectation of a random variable H, defining the number of all horn-trees of order $\leq \gamma(n)$ that are in the giant component. Let HT be such a horn-tree and denote its horn by h. Then $\mathrm{HT} \setminus \{h\}$ must be an isolated tree in $G_p \setminus \{h\}$ and the vertex h must belong to the giant component of $G_p \setminus (\mathrm{HT} \setminus \{h\})$. These two events are independent and the probability of the last one, say $r_{n,k}$, tends to $1 - t(c)$ as $n \to \infty$, where $t(c)$ is defined by (1) (see [2, p. 138]). Consequently, with $E(X_k)$ given by (7),

$$E(H) = \sum_{k=2}^{\gamma(n)} E(X_k)(n-k)\binom{k}{2} p^2 (1-p)^{k(n-k-1)+(k-2)} r_{n,k}$$
$$\to \frac{c}{2}(1-t(c)) \sum_{k=2}^{\gamma(n)} \frac{(ce^{-c})^k}{(k-2)!} k^{k-2}.$$

The last series converges, so denote its sum by $\lambda(c)$. Standard arguments may now be used to show that H has asymptotically Poisson distribution with parameter $\lambda(c)$ (see e.g. [2]). □

The above considerations give a new proof of a conjecture from [3] that a.e. G_p with $p = c/n$, $c > 1$, has an induced tree on $\varphi(c)n$ vertices where $\varphi(c) > 0$ is a function on c. This conjecture was confirmed independently and contemporaneously by Fernandez de la Vega [4] and Frieze and Jackson [5]. In both papers the authors applied sophisticated algorithms constructing an induced tree in G_p and gave their probabilistic analyses which are not straightforward calculations. The advantage of our approach is the much simpler way to succeed in proving the conjecture. However, in a case when $c > 1$ is small our result is weaker than the ones proved in [4] and [5]. For example, it was shown in [4] that a.e. G_p contains a tree on at least $\lfloor \beta(c)n \rfloor$ vertices where $\beta(c)$ is the positive root of the equation

$$c\beta = \log(1 + c^2\beta). \tag{10}$$

One can deduce from this that for $p = 2.45/n$ the order $T_n(p)$ of the largest tree contained in G_p is at least $\lfloor 0.6476n \rfloor$. This is the best lower bound of $T_n(p)$ obtained until this time. On the other hand, taking into account the number of trees on $k = \lfloor \delta n \rfloor$ vertices, by (7), (8) and (9) one can check that

$$\begin{aligned} E(X_k) &= O\left\{ n^{-\frac{3}{2}} \left[\frac{(1-\delta)^{-(1-\delta)} c^{\delta}}{\exp(\delta^2 c/2)} \right]^n \right\} \\ &= o(1) \end{aligned}$$

provided $c = 2.45$ and $\delta \geq 0.92$. Consequently we see that for a.e. G_p with $p = 2.45/n$

$$0.6476n < T_n(p) < 0.92n.$$

Observe also that for large values of c both $\alpha(c)$ and $\beta(c)$ are equal to $\log c/c(1 + o_c(1))$ where $o_c(1)$ is a constant arbitrarily small for c large enough (compare this with Theorem 7). Let us remark here that the results about large induced trees were strengthened by Frieze and Jackson [6] by specifying that the induced tree can be taken to be a path.

The next result deals with the orders of all possible maximal trees in the whole random graph $G_{c/n}$.

Theorem 6. *Let $p = c/n$ where $c > 1$ is a constant, a be given by* (3) *and $\gamma(n) \to \infty$.*

(i) *A.e. G_p has property $\mathcal{T}_k$ for every k satisfying*

$$1 \leq k \leq \left\lfloor \frac{1}{a} (\log n - \tfrac{5}{2} \log \log n - \gamma(n)) \right\rfloor$$

and for some $k \geq \beta(c)n$ where $\beta(c) > 0$ is the root of equation (10).

(ii) *Almost no G_p has property $\mathcal{T}_k$ for every k satisfying*

$$\frac{1}{a} (\log n - \tfrac{5}{2} \log \log n + \gamma(n)) \leq k \leq \alpha(c)n$$

where $\alpha(c)$ is given by (5).

Proof. (i) Denote by Z_k the number of isolated trees of order k in a random graph G_p. Then (see e.g. [1])

$$E(Z_k) \sim nk^{k-2}c^{k-1}e^{-kc}/k!$$

and

$$\mathrm{Var}(Z_k) \sim E(Z_k)\{1 + (c-1)(kc)^{k-1}e^{-kc}/k!\}.$$

Suppose

$$1 \leq k \leq k_0 = \left\lfloor \frac{1}{a} (\log n - \tfrac{5}{2} \log \log n - \gamma(n)) \right\rfloor.$$

Then clearly

$$E(Z_k) \sim \frac{n}{c\sqrt{2\pi}\,k^{\frac{5}{2}}}(ce^{1-c})^k$$

and by Chebyshev's inequality we obtain

$$P(Z_k = 0) \leq \frac{1}{E(Z_k)} + O\left(\frac{k}{n}\right).$$

Consequently

$$\begin{aligned} P(Z_k = 0 \text{ for some } k \leq k_0) &= O\left\{\frac{1}{n}\sum_{k=1}^{k_0} k^{\frac{5}{2}}e^{ak}\right\} \\ &= O\left\{\frac{1}{n}\sum_{i=0}^{\infty} k_0^{\frac{5}{2}}e^{ak_0}e^{-ai}\right\} \\ &= O(e^{-\gamma(n)}) \\ &= o(1). \end{aligned}$$

Thus a.e. G_p has an isolated tree (which is obviously a maximal tree) of order k for every $k \leq k_0$.

The second part of (i) follows from Fernandez de la Vega result [4].

(ii) This is implied by (2) and Theorem 3. □

Finally, let us assume that $p = \omega(n)/n = o(1)$ where $\omega(n)$ is a sequence tending to infinity as $n \to \infty$. Then (see e.g. [2, p. 137]) a.e. G_p is such that every component of G_p, with the exception of its giant component, is a tree of order $o(\log n)$. Clearly, the number of tree components is a monotone decreasing function when $\omega(n)$ is increasing. Eventually, when $\omega(n) = \log n + \gamma(n)$ where $\gamma(n) \to \infty$ the last isolated vertices disappear and a.e. graph G_p becomes connected (see e.g. [2, p. 150]). In our next result we examine orders of maximal trees in the giant component of G_p with $p = \omega(n)/n$. Of course, if $\omega(n) \geq \log n + \gamma(n)$ then by the 'giant component' we mean the whole graph G_p. For the sake of simplicity let us put

$$f(n) = \frac{\log \omega(n)}{\omega(n)}. \tag{11}$$

We have

Theorem 7. *Let $p = \omega(n)/n = o(1)$ and $\varepsilon > 0$ be a constant. Then a.e. G_p is such that every maximal tree in its giant component has the order k, where*

$$(1-\varepsilon)nf(n) < k < (2+\varepsilon)nf(n).$$

Proof. The fact that there is no maximal tree on k vertices for every k, where $1 \leq k \leq (1-\varepsilon)nf(n)$ follows the same lines as the proof of Theorem 3. On the

other hand, by (7) we have

$$
\begin{aligned}
E(X_k) &\leq \frac{1}{\sqrt{2\pi k}}\left(\frac{ne}{k}\right)^k k^{k-2}\left(\frac{\omega(n)}{n}\right)^{k-1} \exp[-pk^2/2] \\
&= O\left\{\frac{n}{k^{5/2}\omega(n)}\left(\frac{e}{\omega(n)^{\varepsilon/2}}\right)^k\right\} \\
&= o(1)
\end{aligned}
$$

provided $k = \lfloor (2+\varepsilon)nf(n) \rfloor$. This implies that there is no tree of order greater than $(2+\varepsilon)nf(n)$ and the proof is completed. □

Having in mind the global structure of a random graph G_p and Theorem 7 we can deduce immediately the following result about $T_n(p)$, the order of the largest tree of G_p.

Corollary 8. *Let $p = \omega(n)/n = o(1)$ and $\varepsilon > 0$ be a constant. Then for a.e. graph G_p*

$$(1-\varepsilon)nf(n) < T_n(p) < (2+\varepsilon)nf(n).$$

The above is a generalization of a result shown in [8] where a case $\omega(n) = e \cdot \log n$ was considered. Also our method of proof is much easier than an algorithmic approach applied in [8].

Let us notice here that in a case when p is fixed then (see [3]) the sequence $\{T_n(p)\}$ of random variables satisfies

$$\frac{T_n(p)}{\log n} \to \frac{2}{\log 1/q} \qquad \text{as } n \to \infty \tag{12}$$

in probability, where $d = 1/q$. (Even a stronger result is true, namely (12) holds with probability one and in any mean, see [7] or [9]). It is a strong evidence to presume that a result in a similar vain is also correct for a wide range of the edge probability p. Thus we state the following

Conjecture 1. Let $p = \omega(n)/n = o(1)$ where $\omega(n) \to \infty$. Then the sequence $\{T_n(p)\}$ of random variables satisfies

$$\frac{T_n(p)}{nf(n)} \to 2 \qquad \text{as } n \to \infty$$

in probability, where $f(n)$ is given by (11).

Also we have found reason to conjecture that every integer between $(1+\varepsilon)nf(n)$ and $(2-\varepsilon)nf(n)$ is likely to occur as the order of a maximal tree of a graph G_p with $p = \omega(n)/n$ (i.e. we would have a similar property of G_p as in the case when p is fixed—see Theorem 1). Thus our second conjecture is as follows.

Conjecture 2. Assume that $p = \omega(n)/n = o(1)$ where $\omega(n) \to \infty$. Then a.e. graph G_p has property $\mathcal{T}_k$ for every k satisfying

$$(1+\varepsilon)nf(n) < k < (2-\varepsilon)nf(n)$$

with $f(n)$ defined by (11).

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Note added in proof

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GENERALIZATIONS OF CRITICAL CONNECTIVITY OF GRAPHS

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We generalize the concepts of a fragment and an atom of a graph and show that these generalizations have properties similar to the common concepts. We prove that a contraction-critical, finite graph G has at least $|G|/3$ triangles and that a finite graph G is 8-connected if every complete subgraph of G is contained in a smallest separating set of G. We study some further classes of graphs (almost critical graphs, C_k-critical graphs) and discuss some applications.

0. Introduction

Properties of atoms and ends derived in [7] have turned out to be valuable tools in studying connectivity of graphs. But sometimes it is inappropriate to consider all smallest separating vertex sets and it is necessary to confine oneself to separating sets having special properties. So we will generalize these concepts in Section 1 and point out which properties of fragments, ends, atoms, and critical graphs carry over to these generalized concepts.

In Sections 2 and 3 we will give some applications of the general results of Section 1. Especially, we shall see in Section 2 that some of the results of [2] are immediate consequences of the properties of $\mathfrak{S}$-atoms and $\mathfrak{S}$-ends. Moreover, we shall prove that *every contraction-critical, finite graph G has at least $\frac{1}{3}|G|$ triangles,* where $|G|$ denotes the number of vertices of G. At the end of the paper we shall see that a contraction-critical, infinite graph does not necessarily contain a triangle, but it must contain one if it is locally finite. In Section 3 we consider C-critical graphs, i.e. graphs where every complete subgraph is contained in a smallest separating vertex set. We shall show that *every C-critical, finite graph is 8-connected,* but I do not yet know if such a graph exists. If there were no C-critical, finite graph this would prove a well-known conjecture of Slater [13].

First we give some definitions and notation. *All graphs* considered in this paper are supposed to be *finite*, unless otherwise stated. The edge joining the vertices x and y is denoted by $[x, y]$, the degree of x in G by $d(x; G)$, and the subgraph of G induced by $T \subseteq V(G)$ by $G(T)$. For $e = [x, y] \in E(G)$, $V(e) := \{x, y\}$ and for

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$T \subseteq V(G)$,

$$N(T;G) := \bigcup_{e \in E(G) \wedge V(e) \cap T \neq \emptyset} V(e) - T.$$

The set of components of G is denoted by $\mathfrak{C}(G)$. For a subgraph $A \subseteq G$, $G - A := G - V(A)$ and also for $V' \subseteq V(G)$, $V' - A := V' - V(A)$. If no confusion seems possible, in set theoretic notation and in notation such as $N(T;G)$ we do not always distinguish strictly between a subgraph and its vertex set. $\mathbb{Z}_n$ means the integers mod n.

We write $\mu(G)$ for the connectivity number of G. We call a separating vertex set T with $|T| = \mu(G)$ a *smallest separating set* of G and define $\mathfrak{T}(G) := \{G(T): T$ a smallest separating set of $G\}$. A *vertex* x of G is called *critical* if $\mu(G - x) = \mu(G) - 1$. Setting $\mathrm{Cr}(G) := \{x \in V(G): x$ critical in $G\}$, we call a *graph* G *critical* if $\mathrm{Cr}(G) = V(G)$. For $T \in \mathfrak{T}(G)$, *a* T*-fragment* is a union of at least one, but not all components of $G - T$. A *fragment* F of G is a T-fragment for any $T \in \mathfrak{T}(G)$, and this uniquely determined $T = G(N(F;G))$ is denoted by T_F. If F is a fragment of G, then $\bar{F} := G - (F \cup T_F)$ is a T_F-fragment, too, the *complementary fragment of* F. (Sometimes we write more exactly $\bar{F}^G$.) An inclusion-minimal fragment is called an *end* and one of least order an *atom*. The order of an atom of G is denoted by $a(G)$.

1. General results

In some proofs in Sections 2 and 3 we do not consider T-fragments for all $T \in \mathfrak{T}(G)$, but only for certain ones. So we need some generalizations of the results on atoms and ends as proved in [7] and [11]. First we give the necessary definitions.

For a graph G, let $\mathfrak{S}$ be a non-empty set of subset of $V(G)$ and define $\mathfrak{T}_{\mathfrak{S}}(G) := \{T \in \mathfrak{T}(G):$ there is an $S \in \mathfrak{S}$ with $S \subseteq T\}$. An $\mathfrak{S}$*-fragment* of G is a T-fragment of G for any $T \in \mathfrak{T}_{\mathfrak{S}}(G)$. An inclusion-minimal $\mathfrak{S}$-fragment of G is called an $\mathfrak{S}$*-end* of G and one of the least vertex numbers is an $\mathfrak{S}$*-atom*. The order of an $\mathfrak{S}$-atom of G is denoted by $a_{\mathfrak{S}}(G)$. The graph G is called $\mathfrak{S}$*-critical* if and only if every $S \in \mathfrak{S}$ is contained in any $T \in \mathfrak{T}(G)$ and for every $\mathfrak{S}$-fragment F, there is a $T \in \mathfrak{T}(G)$ such that $T \cap F \neq \emptyset$ and $T \cap (F \cup T_F)$ does contain an $S \in \mathfrak{S}$.

We give some remarks and examples. Unlike as in the normal use of 'critical', an $\mathfrak{S}$-critical graph is never complete. Hence it has $\mathfrak{S}$-ends and at least one $\mathfrak{S}$-atom. Of course, an $\mathfrak{S}$-atom is an $\mathfrak{S}$-end and an $\mathfrak{S}$-end is connected. If G is $\mathfrak{S}$-critical and $\bigcup_{S \in \mathfrak{S}} S = V(G)$, then G is critical. Usually, $\mathfrak{S}$ will have the property that for every $\mathfrak{S}$-fragment F, there is an $S \in \mathfrak{S}$ such that $S \subseteq F \cup T_F$ and $S \cap F \neq \emptyset$. For an $\mathfrak{S}$ with this property, G is $\mathfrak{S}$-critical if and only if every $S \in \mathfrak{S}$ is in a $T \in \mathfrak{T}(G)$.

Example I.

(1) For any graph G and any positive integer $k \leqslant |G|$, define $\mathfrak{S} := \{S \subseteq V(G) : |S| = k\}$. Then the graph G is $\mathfrak{S}$-critical if and only if every set of k vertices of G is contained in a smallest separating set of G. Such graphs, as well as complete graphs K_n of order $n > k$, are called *k-critical*. Hence a graph G is *k-critical* iff $|G| > k$ and $\mu(G - S) = \mu(G) - k$ for all $S \subseteq V(G)$ with $|S| = k$. An *(n, k)-graph* is a k-critical graph of connectivity number n.

(2) For $\mathfrak{S} := \{\emptyset\}$, an $\mathfrak{S}$-critical graph is called *almost critical*. Including the complete graphs, we define a graph G to be *almost critical* iff $F \cap \mathrm{Cr}(G) \neq \emptyset$ for every fragment F of G. Of course, every critical graph is almost critical, but not vice versa (cf. Example II). A non-complete, almost critical graph is 2-connected (see Corollary 1).

(3) For a graph G and a positive integer k, define $C_k := \{V(K) : K$ is a complete subgraph of G with $|K| = k$ or a clique of G with $|K| \leqslant k\}$. (As usual, a clique is an inclusion-maximal complete subgraph.) Then the graph G is C_k-critical iff every complete subgraph K of G with $|K| \leqslant k$ is contained in a $T \in \mathfrak{T}(G)$. Hence a non-complete graph is C_1-critical iff it is critical, and it is C_2-critical iff it is contraction-critical.

(4) For a graph G, define $C := C_k$ for any integer $k \geqslant |G|$ where C_k is as in (3). Then the graph G is C-critical iff every complete subgraph K of G is contained in a $T \in \mathfrak{T}(G)$. I have not yet found a C-critical finite graph.

(5) For any connected graph G and any positive integer $k \leqslant |G|$, consider $\mathfrak{S} := \{S \subseteq V(G) : |S| = k$ and $G(S)$ connected$\}$. Then G is $\mathfrak{S}$-critical iff every connected subgraph of order k is contained in a $T \in \mathfrak{T}(G)$. Hence this is also a reasonable generalization of 'critical' and 'contraction-critical'.

Before we consider some of the classes given in I in more detail in Sections 2 and 3, we will point out in this section which results transfer to the general case.

Lemma 1 (cf. Lemma 1 in [11]). *Assume $T_i \in \mathfrak{T}(G)$ for $i = 1, 2$ and let F_i be a T_i-fragment of G for $i = 1, 2$ such that $T_1 \cap (F_2 \cup T_2) \cup T_2 \cap (F_1 \cup T_1)$ contains an $S \in \mathfrak{S}$. Then the following statements hold.*

(a) *Assume $F_1 \cap F_2 \neq \emptyset$. Then $|F_1 \cap T_2| \geqslant |\bar{F}_2 \cap T_1|$ and if $F_1 \cap F_2$ is no $\mathfrak{S}$-fragment, $|F_1 \cap T_2| > |\bar{F}_2 \cap T_1|$ holds.*

(b) *Assume $F_1 \cap F_2 \neq \emptyset$. Then $\bar{F}_1 \cap \bar{F}_2 = \emptyset$ or $F_1 \cap F_2$ is an $\mathfrak{S}$-fragment and $\bar{F}_1 \cap \bar{F}_2$ is a fragment of G.*

(c) *If F_2 is an $\mathfrak{S}$-end of G such that $F_2 \nsubseteq F_1$ and $\bar{F}_1 \cap \bar{F}_2 \neq \emptyset$, then $F_1 \cap F_2 = \emptyset$.*

For the proof of Lemma 1 we remark only that in case $F_1 \cap F_2 \neq \emptyset$, $T := F_1 \cap T_2 \cup T_1 \cap T_2 \cup F_2 \cap T_1 \supseteq N(F_1 \cap F_2; G)$ is a separating set containing an $S \in \mathfrak{S}$, hence $G(T) \in \mathfrak{T}_{\mathfrak{S}}(G)$ or $|T| > \mu(G)$. From this, Lemma 1 follows in the same way as Lemma 1 in [11]. □

The proof of the following two lemmas also is a word-for-word translation into the general case of the proof of the corresponding results in [11].

Lemma 2 (cf. Lemma 2 in [11]). *Let B be an $\mathfrak{S}$-end of G and assume there is a $T \in \mathfrak{T}(G)$ such that $T \cap B \neq \emptyset$ and $T \cap (B \cup T_B)$ contains an $S \in \mathfrak{S}$. Then the following statements hold.*

(a) *If there is a T-fragment F of G with $F \cap \bar{B} \neq \emptyset$, then $B \subseteq T$ and $|B| \leq \frac{1}{2}\mu(G)$ or $\bar{F} \subseteq T_B$ and $|\bar{F}| < \frac{1}{2}\mu(G)$.*

(b) *$B \subseteq T$ or $\bar{B} \subseteq T$ holds or there is a T-fragment $F \subseteq T_B$.*

Lemma 3 (cf. Lemma 3 in [11]). *Assume there is an $\mathfrak{S}$-end B of G such that $|B| > \frac{1}{2}\mu(G)$ and $|\bar{B}| \geq \frac{1}{2}\mu(G)$ and there is a $T \in \mathfrak{T}(G)$ such that $T \cap B \neq \emptyset$ and $T \cap (B \cup T_B)$ contains an $S \in \mathfrak{S}$. Then there is a T-fragment F such that $F \subseteq T_B$ and $|F| < \frac{1}{2}\mu(G)$.*

Of course, the T-fragments F, $\bar{F}$ in Lemmata 2 and 3 are $\mathfrak{S}$-fragments, as $T \in \mathfrak{T}_{\mathfrak{S}}(G)$. We need still another simple property of fragments.

Lemma 4. *Let F_i be a T_i-fragment of G for $i = 1, 2$ and assume $F_1 \subseteq T_2$. Then $|F_2 \cap T_1| \geq |F_1|$ or $F_2 \subseteq T_1$. If $T_i \in \mathfrak{T}_{\mathfrak{S}}(G)$ and $|F_1| = a_{\mathfrak{S}}(G)$, then $|F_2 \cap T_1| \geq |F_1|$.*

Proof. If $F_2 \nsubseteq T_1$, then $F_2 \cap \bar{F}_1 \neq \emptyset$ and by Lemma 1(a), $|F_2 \cap T_1| \geq |F_1 \cap T_2| = |F_1|$. For the second assertion, $|F_2| \geq |F_1|$ by definition of $a_{\mathfrak{S}}(G)$ and hence in any case $|F_2 \cap T_1| \geq |F_1|$. □

In a similar way as Theorem 1 in [7] follows, Lemmata 1(a), 1(b), and (4) imply

Theorem 1. *Let A be an $\mathfrak{S}$-atom of G and assume there is a $T \in \mathfrak{T}(G)$ with $T \cap A \neq \emptyset$ such that $T \cap (A \cup T_A)$ contains an $S \in \mathfrak{S}$. Then $A \subseteq T$ and $|A| \leq \frac{1}{2}|T - T_A|$.*

Corollary 1. *For every $\mathfrak{S}$-critical graph G, $a_{\mathfrak{S}}(G) \leq \frac{1}{2}\mu(G)$ holds. In particular, every $\mathfrak{S}$-critical graph is 2-connected.*

It should be emphasized that even in an $\mathfrak{S}$-critical graph G, an $\mathfrak{S}$-atom A does not necessarily have the following property:

$$T \in \mathfrak{T}_{\mathfrak{S}}(G) \quad \text{and} \quad T \cap A \neq \emptyset \rightarrow A \subseteq T.$$

It is not enough that there is an $S \in \mathfrak{S}$ in T, but there must be an $S \in \mathfrak{S}$ in $T \cap (A \cup T_A)$! So it may happen that there are $\mathfrak{S}$-fragments F with $F \cap A \neq \emptyset$, but $A \nsubseteq F$.

We search now for a generalization of Corollary 1 of Theorem 1 in [11], which

says that every non-complete critical graph G has two disjoint fragments of order at most $\frac{1}{2}\mu(G)$. The direct analogue is not true for $\mathfrak{S}$-critical graphs. Let us look at an example.

Example II. Choose any integers $m \geq 1$ and $k \geq 3$. Let $G(2i)$ be a K_{2m+1} for $i = 1, \ldots, k-1$, $G(2i+1)$ a K_{m+1} for $i = 1, \ldots, k-2$, $G(1)$ and $G(2k-1)$ a K_m, and $G(0)$ a K_{m-1} such that $G(i) \cap G(j) = \emptyset$ for $i \neq j$. The graph $G_{m,k}$ arises from $\bigcup_{i=0}^{2k-1} G(i)$ by adding all edges between $G(i)$ and $G(i+1)$ for all $i \in \mathbb{Z}_{2k}$. Then $\mu(G_{m,k}) = 2m$ and $\mathfrak{T}(G_{m,k}) = \{G(0) \cup G(2i+1) : i = 1, \ldots, k-2\} \cup \{G(1) \cup G(2k-1)\}$. Hence $G_{m,k}$ is almost critical, but not critical, and $G(0)$ is the only fragment of $G_{m,k}$ of order at most m.

For some classes of $\mathfrak{S}$-critical graphs we can get disjoint small fragments from

Theorem 2. *Let A be an $\mathfrak{S}$-atom of an $\mathfrak{S}$-critical graph G. Then there is an $\mathfrak{S}$-fragment F disjoint to A and an $R \subseteq V(F)$ with $|R| \leq \frac{1}{2}\mu(G)$ such that there is no $S \in \mathfrak{S}$ in $F \cup N(F-R; G)$ with $S \cap (F-R) \neq \emptyset$.*

Proof. First we remark that an $\mathfrak{S}$-fragment F disjoint from A satisfies the condition of Theorem 2 for $R := V(F)$ if $|F| \leq \frac{1}{2}n$, where $n := \mu(G)$.

Let $B \subseteq \bar{A}$ be an $\mathfrak{S}$-end. Define $\mathfrak{T}' := \{T \in \mathfrak{T}(G) : T \cap B \neq \emptyset$ and there is an $S \in \mathfrak{S}$ in $T \cap (B \cup T_B)\}$ and $\mathfrak{S}' := \{S \in \mathfrak{S} : S \subseteq T \cap (B \cup T_B)$ for a $T \in \mathfrak{T}'\}$. Since G is $\mathfrak{S}$-critical, $\mathfrak{T}' \neq \emptyset$ and $\mathfrak{S}' \neq \emptyset$. Consider any $T \in \mathfrak{T}' \subseteq \mathfrak{T}_{\mathfrak{S}}(G)$ and suppose there is a T-fragment F with $F \cap \bar{B} \neq \emptyset$. Then by Lemma 2(a), B or $\bar{F}$ has order at most $\frac{1}{2}n$ and is disjoint from A. So we may assume $\bar{B} \subseteq T$ for all $T \in \mathfrak{T}' \neq \emptyset$. Hence $\mu(G') = n - |\bar{B}|$ for $G' := G - V(\bar{B})$.

Let B' be an $\mathfrak{S}'$-end of G'. We may assume $|B'| > \frac{1}{2}\mu(G')$ and $|\overline{B'}^{G'}| > \frac{1}{2}\mu(G')$, since B' and $\overline{B'}^{G'}$ are also $\mathfrak{S}$-fragments of G, which are disjoint from A, because $A \subseteq \bar{B}$. As G' has disjoint $\mathfrak{S}'$-ends, we may assume $|B' \cap T_B| \leq \frac{1}{2}n$. Define $R := V(B' \cap T_B)$ and suppose B' and R do not have the properties desired. Then there is an $S \in \mathfrak{S}$ such that $S \subseteq B' \cup N(B'-R; G)$ and $S \cap (B'-R) \neq \emptyset$. This implies $S \subseteq B \cup T_B$ and $S \cap B \neq \emptyset$. G being $\mathfrak{S}$-critical, there is a $T \in \mathfrak{T}(G)$ containing S. Hence $T \in \mathfrak{T}'$ and $S \in \mathfrak{S}'$. Then $T' := T - \bar{B} \in \mathfrak{T}(G')$ satisfies $T' \cap B' \neq \emptyset$ and $S \subseteq T' \cap (B' \cup N(B'; G'))$, since $N(B'-R; G) = N(B'-R; G')$. Hence we can apply Lemma 3 to G', $\mathfrak{S}'$, B', T', S and get a T'-fragment F of G' of order less than $\frac{1}{2}\mu(G')$. Then F is an $\mathfrak{S}$-fragment of G of order at most $\frac{1}{2}n$, which is disjoint from A. $\square$

Corollary 2. *Every C_k-critical graph G has two disjoint C_k-fragments of order at most $\frac{1}{2}\mu(G)$.*

Proof. Let A be a C_k-atom and consider a C_k-fragment F and $R \subseteq V(F)$ as in Theorem 2. Then $F - R = \emptyset$, as every complete graph K with $K \cap (F-R) \neq \emptyset$ is

contained in $F \cup N(F-R; G)$. Hence $|F| \leq \frac{1}{2}\mu(G)$ and by Corollary 1, $|A| \leq \frac{1}{2}\mu(G)$, as well. $\square$

Corollary 3. *Every almost critical, non-complete graph G has disjoint fragments F_1 and F_2 such that $|F_1| \leq \frac{1}{2}\mu(G)$ and $|F_2 \cap \mathrm{Cr}(G)| \leq \frac{1}{2}\mu(G)$.*

Proof. An almost critical, non-complete graph G is $\mathfrak{S}$-critical for $\mathfrak{S} := \{\{x\} : x \in \mathrm{Cr}(G)\}$. Let A be an $\mathfrak{S}$-atom of G and consider an $\mathfrak{S}$-fragment F and $R \subseteq V(F)$ as in Theorem 2. Then $\mathrm{Cr}(G) \cap (F-R) = \emptyset$ and hence $|\mathrm{Cr}(G) \cap F| \leq |R| \leq \frac{1}{2}\mu(G)$. $\square$

2. Contraction-critical and almost critical graphs

Thomassen proved in [14] that *every contraction-critical graph does contain a triangle.* His proof is more or less a proof of Corollary 1 for $\mathfrak{S} := C_2$, which immediately implies this fact. For let A be a C_2-atom of G of order at most $\frac{1}{2}\mu(G)$. If $|A| = 1$, G contains a triangle, as T_A contains an edge. If $|A| \geq 2$, then every edge of A is contained in a triangle of G. This is implied by the following easy lemma, which is used in [14] and explicitly proved in [2].

Lemma 5. *If a fragment F of G has an edge which is not contained in a triangle of G, then $|F| \geq \mu(G)$.*

Proof. If $[x, y] \in E(F)$ is not in a triangle, then $N(x; G) \cap N(y; G) = \emptyset$ and $N(x, G) \cup N(y; G) \subseteq F \cup T_F$. $\square$

The result of Thomassen means that every non-trivial graph G without triangles has an edge which is not in any $T \in \mathfrak{T}(G)$. Such an edge $[x, y] \in E(G) - \bigcup_{T \in \mathfrak{T}(G)} E(T)$ of a non-complete graph G is called contractible. It was proved by Egawa et al. in [2] that *every graph G of connectivity number $n \geq 2$ and of order at least $3n$ without triangles has at least $|G| + \frac{3}{2}n^2 - 3n$ contractible edges.* An essential part of their proof follows immediately from Lemma 3. To see this, consider a graph G of connectivity number $n \geq 2$ without triangles and define $\mathfrak{S} := \{V(e) : e \in \bigcup_{T \in \mathfrak{T}(G)} E(T)\}$. We may assume $\mathfrak{S} \neq \emptyset$. Every $\mathfrak{S}$-fragment F of G has order at least n by Lemma 5, as $|F| = 1$ is impossible. Consider any $\mathfrak{S}$-end B of G. Then $|B| \geq n$ and $|\bar{B}| \geq n$ holds and by Lemma 3 there is no $T \in \mathfrak{T}(G)$ with $T \cap B \neq \emptyset$ such that $T \cap (B \cup T_B)$ contains an $S \in \mathfrak{S}$. Hence every $e \in E(G)$ with $V(e) \cap B \neq \emptyset$ is contractible. Choosing an $\mathfrak{S}$-end $B' \subseteq \bar{B}$, we get disjoint ends B and $B' \subseteq \bar{B}$ such that every edge incident to a vertex of $B \cup B'$ is contractible. This result was an important step in the proof given by Egawa et al. Another crucial step in their proof was the fact that every vertex z of G is incident to at least two contractible edges. To see this, define $\mathfrak{S}_z :=$

$\{\{z,x\}:[z,x]\in\bigcup_{T\in\mathfrak{T}(G)}E(T)\}$, assume $\mathfrak{S}_z\neq\emptyset$, and consider an $\mathfrak{S}_z$-end B. Then Lemma 3 implies again that all $[z,x]\in E(G)$ with $x\in B$ are contractible. As there is at least one such edge (because $z\in T_B$) and there is an $\mathfrak{S}_z$-end $B'\subseteq\bar{B}$, we get two contractible edges in z.

We study now the number and distribution of triangles in contraction-critical graphs. We shall show that *every C_2-critical graph G has at least $\frac{1}{3}|G|$ triangles* and that *every C_2-fragment of G contains a vertex which is on a triangle of G.*

Theorem 3. *Let G be a C_2-critical graph and assume $z\in V(G)$ is not contained in a triangle of G. Then there are an $x\in N(z;G)$ and an edge $[x,y]\in E(G)$ such that there at least $d(x;G)-\frac{1}{2}(\mu(G)-1)$ triangles containing $[x,y]$.*

Proof. Denote $n:=\mu(G)$ and define $\mathfrak{S}:=\{\{z,x\}:[z,x]\in E(G)\}$. As every $\mathfrak{S}$-fragment F has $F\cap N(z;G)\neq\emptyset$, the graph G is $\mathfrak{S}$-critical, because it is C_2-critical. Let us consider an $\mathfrak{S}$-atom A. Applying Theorem 1, we get $|A|\leqslant\frac{1}{2}(n-1)$, since $z\in T$ for all $T\in\mathfrak{T}_{\mathfrak{S}}(G)$. As z is not on a triangle, but $T_A\in\mathfrak{T}_{\mathfrak{S}}(G)$, we have $|A|\geqslant 2$. Hence there is an $[x,y]\in E(A)$ with $x\in N(z;G)$. Denoting $c:=|N(x;G)\cap N(y;G)|$, we get the inequalities $d(x;G)-c+d(y;G)-c+c\leqslant|A|+|T_A|\leqslant\frac{1}{2}(3n-1)$, which implies the claim that $c\geqslant d(x,G)-\frac{1}{2}(n-1)$. $\square$

Remark. For $n=4$, we get $|A|=1$ for the $\mathfrak{S}$-atom A in the proof of Theorem 3. Hence in a C_2-critical graph of connectivity number 4, every vertex z is on a triangle which contains a vertex $x\neq z$ of degree 4. This was used by Fontet in [3] and [4].

Theorem 4. *Every C_2-critical graph G has at least $\frac{1}{3}|G|$ triangles.*

Proof. Let $t(G)$ denote the number of triangles of G and for $e\in E(G)$, let $t(e)$ be the number of triangles of G containing e. Define $H:=(V(G),\{e\in E(G):t(e)>0\})$ and $J:=\{z\in V(G):d(z;H)=0\}$. For $z\in J$, we choose an $x:=f(z)$ in $N(z;G)$ as in Theorem 3. Denoting $n:=\mu(G)$, then $|f^{-1}(x)|<\frac{1}{2}(n-1)$ for $x\in X:=f(J)$ by Theorem 3 and, therefore, $\sum_{[x,y]\in E(H)}t([x,y])\geqslant 2(d(x;G)-\frac{1}{2}(n-1))\geqslant n+1>2(|f^{-1}(x)|+1)$ for $x\in X$ by Theorem 3. So we get

$$(\alpha)\qquad \sum_{x\in X}\sum_{[x,y]\in E(H)}t([x,y])\geqslant 2(|J|+|X|).$$

Since every $y\in Y:=V(G)-(X\cup J)$ is on a traingle in G, we have also

$$(\beta)\qquad \sum_{y\in Y}\sum_{[y,x]\in E(H)}t([y,x])\geqslant 2|Y|.$$

Addition of (α) and (β) gives the assertion

$$2|G|\leqslant\sum_{x\in V(G)}\sum_{[x,y]\in E(H)}t([x,y])=6t(G).\quad\square$$

One could conjecture that every vertex of a contraction-critical graph is on a triangle. But this is not true, as the following example shows.

Example III. Take any even integer $m \geq 4$ and any integer $k \geq 2m+1$. Consider k disjoint complete graphs K^i of order m, say, $V(K^i) = \{a_1^i, \ldots, a_m^i\}$. The graph $R'_{m,k}$ then arises from $\bigcup_{i=1}^k K^i$ by adding all edges $[a_j^i, a_{j+h}^{i+1}]$ for $h = 1, \ldots, \frac{1}{2}m + 1$, $i \in \mathbb{Z}_k$, and $j \in \mathbb{Z}_m$. As $m \geq 4$, there is an independent vertex set U with $|U| = k$, for instance, $\{a_1^1, a_1^2, \ldots, a_1^k\}$. Construct $R_{m,k}$ from $R'_{m,k}$ by adding any further vertex z and all edges $[z, u]$ for $u \in U$. Then $\mu(R_{m,k}) = 2m+1$ and $R_{m,k}$ is C_2-critical, but there is no triangle containing z.

In Section 3 we will need the fact that every C_2-fragment of a C_2-critical graph G has a vertex contained in a triangle of G. For this we need

Lemma 6. *Let B be an independent vertex set of G such that $A := G - B$ is not independent and $|B| \geq |A| \geq 2$ holds. Assume $d(a; G) \geq |B|$ for all $a \in A$ and there is an $a \in A$ with $d(a; G) > |A|$. Then there is a triangle D in G with $D \cap B \neq \emptyset$.*

Proof. By induction on $|A|$. The claim is obvious for $|A| = 2$. Suppose $|A| > 2$. If A is disconnected, we can delete a least component of A and by induction get the triangle desired. Let us assume that A is connected. By assumption, there is an $a_0 \in V(A)$ with $d(a_0; G) > |A|$. There is an $a \neq a_0$ in A such that $A - \{a\}$ is connected. Since $d(a; G) \geq |A|$, there is a $b \in B \cap N(a; G)$. If there is no triangle containing $[a, b]$, then $G' := G - \{a, b\}$, $B' := B - \{b\}$, and $A' := A - \{a\}$ satisfy the conditions of the lemma. Then, by induction, there is a triangle D with $D \cap B' \neq \emptyset$. □

Theorem 5. *Let F be a C_2-fragment of a C_2-critical graph G. Then there is a triangle D in G with $D \cap F \neq \emptyset$.*

Proof. Choose a C_2-end $B \subseteq F$. Let us suppose there is no triangle in G containing a vertex of B. Then $|B| \geq 2$ as $E(T_B) \neq \emptyset$ and hence $|B| \geq \mu(G) =: n$ by Lemma 5. Define $\mathfrak{S} := \{V(e) : e \in E(G)$ and $V(e) \cap B \neq \emptyset\}$ and let A be an $\mathfrak{S}$-atom of G. Since T_A contains an edge incident to an $x \in B$, we have $|A| \geq 2$ and $N(x; G) \cap A \neq \emptyset$. Then $\{x, z\} \in \mathfrak{S}$ for $z \in N(x; G) \cap A \neq \emptyset$ and Theorem 1 implies $|A| \leq \frac{1}{2}(n-1)$. But then $B \cap A = \emptyset$ by Lemma 5, since $|A| \geq 2$. Hence $B \cap \bar{A} \neq \emptyset$, since $|B| \geq n$. B being a C_2-end, $B \cap \bar{A}$ cannot be a C_2-fragment. Hence Lemma 1(b) and (a) implies $\bar{B} \cap A = \emptyset$ and $|B \cap T_A| > |A \cap T_B| = |A|$. Let us consider the graph $H := G(A \cup (B \cap T_A)) - E(B \cap T_A)$. Then $d(a; H) = d(a; G - (T_A - B)) \geq n - |T_A - B| = |T_A \cap B|$ for all $a \in A$. As $|T_A \cap B| > |A| \geq 2$, we can apply Lemma 6 to H for $V(B \cap T_A)$ and A and get a triangle D intersecting $B \cap T_A$, a contradiction to our assumption. □

We turn now to almost critical graphs. These graphs occur in a natural way; for given an atom A of a contraction-critical graph G, $G-A$ is almost critical. This is generalized in

Lemma 7. *Let A be an $\mathfrak{S}$-atom of an $\mathfrak{S}$-critical graph G. If for every $x \in T_A$, there is an $S \in \mathfrak{S}$ containing x such that $S \cap A \neq \emptyset$ and $S \subseteq A \cup T_A$, then $G - V(A)$ is almost critical and $V(T_A) \subseteq \mathrm{Cr}(G - V(A))$.*

Proof. Consider $G' := G - V(A)$ and any $x \in T_A \neq \emptyset$. There is an $S \in \mathfrak{S}$ such that $S \cap A \neq \emptyset$ and $x \in S \subseteq A \cup T_A$. G being $\mathfrak{S}$-critical, there is a $T \in \mathfrak{T}(G)$ containing S and $A \subseteq T$ by Theorem 1. Hence $\mu(G') = \mu(G) - |A|$ and $x \in \mathrm{Cr}(G')$. As $F' \cap T_A \neq \emptyset$ for every fragment F' of G', G' is almost critical. □

Remark. Instead of the assumption 'A is an $\mathfrak{S}$-atom', it is enough to assume in Lemma 7 that a non-empty $A \subsetneq V(G)$ has the property: If $T \cap A \neq \emptyset$ for a $T \in \mathfrak{T}(G)$ and $T \cap (A \cup N(A;G))$ contains an $S \in \mathfrak{S}$, then $A \subseteq T$. So, for instance, it is true that $G - z$ is almost critical for any vertex z of a contraction-critical graph G.

In Theorem 2 of [11] it was proven that every non-complete, critical graph has four disjoint fragments. This is not true for almost critical graphs, as Example II shows. A still simpler counterexample is the following graph: For any integer $n \geqslant 3$, choose a complete graph K of order at least $2n$, add two adjacent vertices a_1, a_2 and join a_i to a set A_i of exactly $n-1$ vertices of K for $i = 1, 2$, such that $|A_1 \cup A_2| > n$. This graph is almost critical and does not have four disjoint fragments. Analysing the proof of Theorem 2 in [11], it is easily seen that the same arguments even prove the following more general result.

Theorem 6. *Every almost critical, non-complete graph G has fragments F_1, F_2, F_3, F_4 such that F_1, F_2, F_3, and $F_4 \cap \mathrm{Cr}(G)$ are disjoint.*

Since $F \cap \mathrm{Cr}(D) \neq \emptyset$ for a fragment F of G, $F_4 \nsubseteq F_i$ and $F_i \nsubseteq F_4$ for $i = 1, 2, 3$. In Theorem 6, we can choose F_1 as an atom of G and get, in addition, that $F_1 \cap F_4 = \emptyset$. But it is not possible to achieve, moreover, $F_2 \cap F_4 = \emptyset$, as Example II shows.

Egawa proved in [1] that $a(G) \leqslant \frac{1}{4}(\mu(G))$ *for every non-complete, contraction-critical graph G.* Considering an atom of G, this result follows immediately from Lemma 7, Theorem 6, and Lemma 4.

3. C_k-critical graphs

In this section we will study mainly the connectivity of C_k-critical graphs. We shall show that *C_3-critical graphs are 6-connected* and that *every C-critical graph*

G is 8-connected and does not contain a complete graph of order $\mu(G)-2$, if it is minimally $\mu(G)$-connected. Unfortunately, I have not succeeded in deciding if there are finite C-critical graphs. This class of graphs is related to a well-known *conjecture of Slater* [13], which says that *for all positive integers n the only $(n, \lfloor\frac{1}{2}n\rfloor+1)$-graph* (cf. Example I (1)) *is K_{n+1}*. (Some stronger conjectures are found in [10].) Let us verify that there is no non-complete $(n, \lfloor\frac{1}{2}n\rfloor+1)$-graph, if $\mu(G)>n$ for every C-critical graph G. Suppose G is a non-complete $(n, \lfloor\frac{1}{2}n\rfloor+1)$-graph. If there is no C-critical graph of connectivity number n, there must be a complete subgraph $K\subseteq G$ not contained in any $T\in\mathfrak{T}(G)$. As G is $(\lfloor\frac{1}{2}n\rfloor+1)$-critical, $|K|>\lfloor\frac{1}{2}n\rfloor+1$. But G cannot contain such a large complete subgraph (see Theorem 1 in [12]). If there were no C-critical graph, therefore, Slater's conjecture would be settled. It is obvious that there is no C-critical line graph $L(G)$, because $\emptyset\neq\{[a,x]:=x\in N(a;G)\}\subseteq T\in\mathfrak{T}(L(G))$ for any $a\in V(G)$ would imply the contradiction that $T-\{[a,b]\}$ is also separating in $L(G)$ for $b\in N(a;G)$. Hence Slater's conjecture is true for line graphs, as proved in [6].–For proving Slater's conjecture, it would be sufficient to show that every k-critical graph does contain a K_{k+1}. This could be true for all k and certainly is true for $k=2$ by Thomassen's result, which we generalized in Section 2. But I do not know if every 3-critical graph must contain a K_4. I do not even know if there is any k such that every k-critical graph must contain a K_4.

First, let us consider examples showing that for every positive integer k there are C_k-critical graphs which are neither C_{k+1}-critical nor 3-critical.

Example IV. For positive integers k and $m>3k$ define $H_m(k):=(\mathbb{Z}_m, \{[x, x+\kappa]: x\in\mathbb{Z}_m \text{ and } \kappa=1,\dots,k\})$. Then $\mu(H_m(k))=2k$ and $H_m(k)$ is C_k-critical, but neither C_{k+1}-critical nor 3-critical. The graphs $H_m(k)$ also show that for given k, there are C_k-critical graphs of connectivity number $2k$ having arbitrarily large order. This is in contrast with the results for 3-critical graphs in [9], where it was proved that every $(n, 3)$-graph has less than $6n^2$ vertices.

For the proof of the fact that every C-critical graph is 8-connected, we need a series of specialized lemmata.

Lemma 8. *Let B be a fragment of a graph G with $V(B)=\{a,b\}$. If $a\in T\in\mathfrak{T}(G)$, then $b\in T$ or $V(T-\bar{B})\subseteq N(b;G)$.*

The easy proof of this lemma is left to the reader.

Lemma 9. *Let F_i be a T_i-fragment of G for $i=1,2$ such that $F_1\cap F_2\neq\emptyset$ and consider $B\subseteq T_1\cap T_2$. Then $|F_1\cap T_2|\geq|\bar{F}_2\cap T_1|+|B|-|N(B;G)\cap F_1\cap F_2|$ or $V(F_1\cap F_2)\subseteq N(B;G)$. If $|N(B;G)\cap F_1\cap F_2|<|B|\leq 2$, then $|F_1\cap T_2|>|\bar{F}_2\cap T_1|$ or $V(F_1\cap F_2)=\{z\}$ with $N(z;G)=V(F_1\cap T_2)\cup V(T_1\cap T_2)\cup V(\bar{F}_2\cap T_1)$.*

Proof. Suppose $R := V(F_1 \cap F_2) - N(B; G) \neq \emptyset$. Then $N(R; G) \subseteq N(B; G) \cap F_1 \cap F_2 \cup F_1 \cap T_2 \cup F_2 \cap T_1 \cup (T_1 \cap T_2 - B)$ and hence $|T_1| = \mu(G) \leq |N(R; G)| \leq |N(B; G) \cap F_1 \cap F_2| + |F_1 \cap T_2| + |F_2 \cap T_1| + |T_1 \cap T_2| - |B|$, which implies the first inequality. If $|N(B; G) \cap F_1 \cap F_2| < |B| \leq 2$ and $|F_1 \cap T_2| = |\bar{F}_2 \cap T_1|$ (cf. Lemma 1(a)), then $V(F_1 \cap F_2) \subseteq N(B; G)$ by the assertion yet proved. Hence $V(F_1 \cap F_2) = \{z\}$ and $|N(z; G)| \leq |F_1 \cap T_2 \cup T_1 \cap T_2 \cup F_2 \cap T_1| = |T_1| = \mu(G)$, proving the second assertion. □

Lemma 10. *Let B be an S-fragment of $G(S \in \mathfrak{T}(G))$ with $|B| \leq 2$, $T \in \mathfrak{T}(G)$ with $B \subseteq T$, and $C \in \mathfrak{C}(G - T)$ such that $|C \cap S| = |B|$ and $|C| \geq 2$. Let K' be a clique of G such that $K' \cap C \cap S \neq \emptyset$ and $K' \cap B \neq \emptyset$, and assume $T' \in \mathfrak{T}(G)$ with $T' \supseteq K'$. Then $B \subseteq T'$ and the following two statements hold.*

(1) *If $C' \cap C \neq \emptyset$ for $C' \in \mathfrak{C}(G - T')$, then $|C' \cap T| > |\bar{C} \cap T'|$ and $|C \cap T'| > |\bar{C}' \cap T|$.*

(2) $|C' \cap T| \geq 2$ *for all* $C' \in \mathfrak{C}(G - T')$, $|C \cap T'| \geq 2$, *and* $|\bar{C} \cap T'| \geq 2$ *or* $|\bar{C} \cap T'| = |\bar{C}| = 1$.

Proof. Suppose $B \nsubseteq T'$. Then $|B| = 2$ and by Lemma 8 we have $N(b; G) \supseteq V(T' - \bar{B})$ for the vertex $b \in B - T'$. But this is impossible, because $T' - \bar{B}$ contains the clique K'. Hence $B \subseteq T$.

Since $C \cap S \cap T' \neq \emptyset$ and $|C \cap S| = |B|$, we have $|C' \cap C \cap S| < |B|$ for all $C' \in \mathfrak{C}(G - T')$. As $T' \cap G(C \cup T)$ contains the clique K', there cannot be a vertex z with $N(z; G) \supseteq V(T' \cap (C \cup T))$. If $C' \cap C \neq \emptyset$ for $C' \in \mathfrak{C}(G - T')$, therefore Lemma 9 implies $|C' \cap T| > |\bar{C} \cap T'|$ and $|C \cap T'| > |\bar{C}' \cap T|$. Hence (1) follows.

Consider any $C' \in \mathfrak{C}(G - T')$. As T' contains a clique, $|C'| \geq 2$. If $C' \cap C \neq \emptyset$, then $C' \cap T \neq \emptyset$ by (1). If $C' \cap C = \emptyset$, then $C' \cap T \neq \emptyset$, because $T' \cap C \supseteq K' \cap C \neq \emptyset$. Hence $C' \cap T \neq \emptyset$ for all $C' \in \mathfrak{C}(G - T')$ and $T' \cap \bar{C} \neq \emptyset$.

For proving $|C \cap T'| \geq 2$, we may assume $C \nsubseteq T'$, because $|C| \geq 2$. Hence there is a $C' \in \mathfrak{C}(G - T')$ with $C' \cap C \neq \emptyset$. But then (1) implies $|C \cap T'| \geq 2$, since $\bar{C}' \cap T \neq \emptyset$.

Consider any $C' \in \mathfrak{C}(G - T')$. We may assume again $C' \nsubseteq T$. If $C' \cap C \neq \emptyset$, (1) implies $|C' \cap T| \geq 2$, as $\bar{C} \cap T' \neq \emptyset$. If $C' \cap \bar{C} \neq \emptyset$, Lemma 1(a) implies $|C' \cap T| \geq |C \cap T'| \geq 2$. Hence $|C' \cap T| \geq 2$ for all $C' \in \mathfrak{C}(G - T')$.

If $|\bar{C}| \geq 2$ and $\bar{C} \nsubseteq T'$, then Lemma 1(a) implies $|\bar{C} \cap T'| \geq 2$, because $|\bar{C}' \cap T| \geq 2$ for all $C' \in \mathfrak{C}(G - T')$. □

Lemma 11. *Let G be a C-critical graph with $\mu(G) \leq 7$ and let B be an S-fragment $(S \in \mathfrak{T}(G))$ isomorphic to K_2. Assume K is a clique in G with $K \cap B \neq \emptyset$ and $K \subseteq T \in \mathfrak{T}(G)$. Then $|T \cap S| \leq 1$.*

Proof. We will obtain a contradiction from the assumption $|T \cap S| \geq 2$.

As $T - \bar{B}$ contains the clique K, $B \subseteq T$ by Lemma 8 and $|C| \geq 2$ for all

$C \in \mathfrak{C}(G-T)$. Hence Lemma 4 implies $|C \cap S| \geq 2$ for all $C \in \mathfrak{C}(G-T)$. Since $|T \cap S| \geq 2$ and $\mu(G) \leq 7$, there is a $C \in \mathfrak{C}(G-T)$ with $|C \cap S| = 2$, say, $V(C \cap S) = \{c, \bar{c}\}$. This notation c, $\bar{c}$ may be chosen in such a way that $N(c; G) \supseteq V(B)$, if there is such a vertex in $C \cap S$. As $c \in V(S) = N(B; G)$, there are cliques K' with $c \in K'$ and $K' \cap B \neq \emptyset$. Choose such a clique K' so that $|K' \cap T \cap S|$ is maximal. G being C-critical, there is a $T' \in \mathfrak{T}(G)$ containing K'. Applying Lemma 10, we get $B \subseteq T'$ and

(a) $|C \cap T'| \geq 2$, $|\bar{C} \cap T'| \geq 2$, *and* $|C' \cap T| \geq 2$ *for all* $C' \in \mathfrak{C}(G-T')$.

Next we prove

(b) *Assume there is a* $C' \in \mathfrak{C}(G-T')$ *with* $C' \cap C \neq \emptyset$. *Then* $|C \cap T'| = |C' \cap T| = 3$, $T \cap T' = B$, *and* $\mu(G) = 7$. *Furthermore,* $\bar{C}' \subseteq T$ *and* $|\bar{C}'| = 2$ *hold.*

From (a) and $T \cap T' \supseteq B$ we get $7 \geq |T| \geq |C' \cap T| + 2 + 2$, hence $|C' \cap T| \leq 3$ and in the same way $|C \cap T'| \leq 3$. On the other hand, (a) and the first part of Lemma 10 imply $|C' \cap T| \geq 3$ and $|C \cap T'| \geq 3$. So $|C' \cap T| = 3 = |C \cap T'|$. From $|C' \cap T| = 3$ and (a) we get $T' \cap T = B$, $|\bar{C}' \cap T| = 2$, and $\mu(G) = 7$. But then $\bar{C}' \cap C = \emptyset$, because $\bar{C}' \cap C \neq \emptyset$ implies $|\bar{C}' \cap T| \geq 3$ in the same way as above for C'. Lemma 1(a) shows $\bar{C}' \cap \bar{C} = \emptyset$, since $|\bar{C}' \cap T| < |C \cap T'|$. Hence $\bar{C}' \subseteq T$.

(c) *If* $N(c; G) \cap T \cap S \neq \emptyset$, *then* $T' \cap T \cap S \neq \emptyset$.

Assume there is an $x \in N(c; G) \cap T \cap S$. If $V(B) \subseteq N(c; G)$, there is a triangle $D \supseteq \{c, x\}$ with $D \cap B \neq \emptyset$. If $V(B) \nsubseteq N(c; G)$, then by the choice of c, also $V(B) \nsubseteq N(\bar{c}'; G)$. But this implies $V(B) \subseteq N(x; G)$ and there is again a triangle D as above. Hence, by the choice of K', $|K' \cap T \cap S| \geq |D \cap T \cap S| = 1$, which implies $|T' \cap T \cap S| \geq 1$.

(d) $C \subseteq T'$ *and* $2 \leq |C| \leq 3$.

Suppose $C \nsubseteq T'$. Then there is a $C' \in \mathfrak{C}(G-T')$ such that $C' \cap C \neq \emptyset$. Then $\bar{C}' \subseteq T$ and $|\bar{C}'| = 2$ by (b) and Lemma 4 implies $|\bar{C}' \cap S| \geq 2$, hence $\bar{C}' \subseteq T \cap S$. From $c \in T'$ we get $N(c; G) \cap \bar{C}' \neq \emptyset$, hence $N(c; G) \cap T \cap S \neq \emptyset$. But this implies $T' \cap T \cap S \neq \emptyset$ by (c), in contradiction to $T' \cap T = B$ which holds by (b).The bounds $2 \leq |C| \leq 3$ follow from (a) and $|T' \cap T| \geq |B| = 2$.

(e) $|C| = 2$.

Suppose $|C| \neq 2$, hence $|C| = 3$ by (d). Then there is at most one $C' \in \mathfrak{C}(G-T')$ with $C' \cap \bar{C} \neq \emptyset$, because $C' \cap \bar{C} \neq \emptyset$ implies $|C' \cap T| \geq |C \cap T'| = 3$ by Lemma 1(a) and (d). Hence there are $C' \in \mathfrak{C}(G-T')$ with $C' \subseteq T$ and then there is even a $C' \in \mathfrak{C}(G-T')$ such that $C' \subseteq T$ and $|C'| = 2$. Then by Lemma 4 again $C' \subseteq T \cap S$ and hence $N(c; G) \cap T \cap S \neq \emptyset$. But then (c) implies $|T \cap T'| \geq |B| + 1 = 3$, hence $7 \geq |C| + |T \cap T'| + |\bar{C} \cap T'| \geq 6 + |\bar{C} \cap T'|$, thus contradicting (a).

(f) $T' \cap T \cap S \neq \emptyset$ and $|T \cap S| = 2$.

Since $|C| = 2$ by (e) and $|T \cap S| \geq 2$ by assumption, $N(c; G) \cap T \cap S \neq \emptyset$ holds. Hence $T' \cap T \cap S \neq \emptyset$ by (c). Now suppose $|T \cap S| \geq 3$. Then $2 \leq |\bar{C} \cap S| \leq 7 - |T \cap S| - |C \cap S| \leq 2$ by Lemma 4. Hence $|\bar{C} \cap S| = 2$ and $\bar{C} \in \mathfrak{C}(G-T)$. But then we can take $\bar{C}$ instead of C and get also $\bar{C} \subseteq T'$ and $|\bar{C}| = 2$ from (d) and (e). As $|C' \cap S| \geq 2$ for all $C' \in \mathfrak{C}(G-T')$ by Lemma 4, we get the contradiction $7 \geq |S| \geq |T' \cap S| + |C' \cap S| + |\bar{C}' \cap S| \geq 4 + 2 + 2$.

Using properties (a) through (f), we can now complete the proof of Lemma 11. Since $|T \cap S| = 2$ and $T' \cap T \cap S \neq \emptyset$ by (f), we have $|C' \cap T \cap S| \leq 1$ for all $C' \in \mathfrak{C}(G - T')$. Hence $C' \nsubseteq T$ and, therefore, $C' \cap \bar{C} \neq \emptyset$ for all $C' \in \mathfrak{C}(G - T')$, as $C \subseteq T'$ by (d). Furthermore, $|C' \cap T| = 2$ for all $C' \in \mathfrak{C}(G - T')$ by (a), as $|T' \cap T| \geq |B| + |T' \cap T \cap S| \geq 3$. Since $|\bar{C} \cap S| \leq 3$, there is a $C' \in \mathfrak{C}(G - T')$ such that $|C' \cap \bar{C} \cap S| \leq 1$. Then $C' \cap T \cap S \neq \emptyset$ and by Lemma 9, $C' \cap \bar{C}$ consists of a single vertex z with $N(z; G) \supseteq V(T - \bar{C}')$. Since $|T \cap S| = 2$, $T' \cap T \cap S \neq \emptyset$, and $C' \cap T \cap S \neq \emptyset$, we get $V(B) \cup V(T \cap S) \subseteq V(T - \bar{C}') \subseteq N(z; G)$, which is impossible, because the clique K is contained in $G(B \cup T \cap S)$. □

Lemma 11 immediately implies

Lemma 12. *Let G be a C-critical graph with $\mu(G) \leq 7$ and let B be a fragment of G isomorphic to K_2. Then there is no triangle D in G such that $|D \cap B| = 1$.*

Now we are prepared to prove the main result of this section.

Theorem 7. *Every finite, C-critical graph G is 8-connected.*

Proof. We assume $\mu(G) \leq 7$. Define $\mathfrak{S} := \{V(K) : K$ clique of G with $|K| \geq 3\}$. By Thomassen's result [14] (cf. theorem 4), $\mathfrak{S} \neq \emptyset$. By Theorem 5, G is $\mathfrak{S}$-critical. Choose an $\mathfrak{S}$-atom A of G and define $S := N(A; G)$. By the definition of $\mathfrak{S}$, there is a triangle $\bar{D}$ in $G(S)$. By Corollary 1, the assumption, and the definition of $\mathfrak{S}$, $2 \leq |A| \leq 3$.

If $|A| = 2$, then there is an $a \in A$ with $|N(a; G) \cap \bar{D}| \geq 2$ and hence there is a triangle D' with $|D' \cap A| = 1$, contradicting Lemma 12. Hence $|A| = 3$.

By Theorem 5, there is a triangle D with $D \cap A \neq \emptyset$. Choose a triangle D with $D \cap A \neq \emptyset$ so that $|D \cap S|$ is as large as possible. Let K be a clique containing D and $T \in \mathfrak{T}(G)$ containing K. Then $T \in \mathfrak{T}_{\mathfrak{S}}(G)$ and $A \subseteq T$ by Theorem 1. So $|C \cap S| \geq 3$ for all $C \in \mathfrak{C}(G - T)$ by Lemma 4. This implies $\mu(G) \geq 6$ and $|T \cap S| \leq 1$. Hence $|D \cap S| \leq 1$ and by the choice of D, there is no triangle D' with $|D' \cap A| = 1$. It is easy to see then that A is a triangle and $G(\bar{D} \cup A) - (E(\bar{D}) \cup E(A))$ consists of three disjoint edges $[d_i, a_i]$, where $V(\bar{D}) = \{d_1, d_2, d_3\}$ and $V(A) = \{a_1, a_2, a_3\}$. Then $S' := (S - \{d_1\}) \cup \{a_1\}$ is a smallest separating set of G and $B := A - \{a_1\}$ is a component of $G - S'$. As $\mu(G) \geq 6$, there is an $s \in S - \bar{D}$. But then s, a_1, a_2 span a triangle D' with $|D' \cap B| = 1$, contradicting Lemma 12. □

Remark. At the end of Section 2 we saw that a result of Egawa [1] follows immediately from Theorem 6. If we had a result similar to Theorem 6 for all $\mathfrak{S}$-critical graphs, (perhaps) it would be possible to derive Theorem 7 from such a result in an analogous way.

In the next theorem we shall prove that *a C-critical, minimally n-connected graph cannot contain a K_{n-2}*. (A graph G is called *minimally n-connected*, if

$\mu(G)=n$, but $\mu(G-e)<n$ for all $e \in E(G)$.) This is not right for C_3-critical, minimally n-connected graphs as the graphs $H_m(3)$ from Example IV show. We give yet another example.

Example V. For integers $m>i\geq 1$, let $G_{m,i}$ arise from $K^1 \cup K^2$ by adding the edges $[x^1_\kappa, x^2_{\kappa+\lambda}]$ for $\lambda=1,\ldots,i$ and $\kappa \in \mathbb{Z}_m$, where K^1 and K^2 are disjoint complete graphs of order m with $V(K^k)=\{x^k_1,\ldots,x^k_m\}$ for $k=1,2$. The graphs $G_{m,i}$ are $(m+i-1)$-regular and $(m+i-1)$-connected. As for $m\geq 2i-1$ and $j\leq i$, every complete subgraph of $G_{m,i}$ of order j is contained in a complete subgraph of order $j+1$, the graphs $G_{m,i}$ are C_i-critical for all $m\geq 2i-1$. Hence, *for all $i\geq 1$ and $n\geq \max(3i-2,2)$, there are C_i-critical, minimally n-connected graphs which do contain a K_{n+1-i}.*

It is well known that a non-complete, minimally n-connected graph cannot contain a K_{n+1} for $n\geq 2$ (cf. [5] or [8]). A C_2-critical, minimally n-connected graph G cannot contain a K_n, because such a subgraph K_n has a vertex z of degree n in G (see [8]) and then for the vertex $x \in N(z;G)-V(K_n)$, the edge $[z,x]$ cannot be in a $T \in \mathfrak{T}(G)$. (It is well known that there are no C_2-critical graphs of connectivity number 2. This follows, for instance, from Theorem 1, which shows that a C_2-atom of a C_2-critical graph G has order at most $\frac{1}{2}(\mu(G)-1)$.) I do not know if a C_3-critical, minimally n-connected graph can contain a K_{n-1}, but this seems unlikely. If minimality is not assumed, there is no upper bound for the order of a complete graph contained in a C_i-critical graph of connectivity number n. This is shown by the following

Example VI. Take any positive integers $k\geq i$ and define $n:=(k+1)i$. Let H be a bipartite graph with bipartition A, B (i.e. A, B is a partition of $V(H)$ into independent vertex sets) and let $B_1,\ldots,B_k$ be a partition of B. For $\kappa=1,\ldots,k$, the graph $H(A\cup B_\kappa)$ may arise from a complete bipartite graph $K_{i+1,i+1}$ with bipartition A, B_κ by deleting the edges of a 1-factor. Then $N(A';H)=B$ for all $A'\subseteq A$ having at least two elements, hence

$$(Z)\ |N(A';H)|=k(i+1)\geq ki+i=n.$$

Define $\bar{H}:=H\cup C$, where C is a complete graph with $V(C)=A$. Then $d(a;\bar{H})=n$ for all $a\in A$. Let K be a complete graph of order at least $k(i+1)$. For $j\in J$, let $\bar{H}_j$ be a copy of $\bar{H}$, where $\bar{A}_j\subseteq V(\bar{H}_j)$ and $\bar{B}_j\subseteq V(\bar{H}_j)$ correspond to A and B, respectively. Assume $\bar{A}_j\cap \bar{A}_{j'}=\emptyset$ for all $j\neq j'$, $\bar{A}_j\cap V(K)=\emptyset$ and $\bar{B}_j\subseteq V(K)$ for all $j\in J$ such that for every $S\subseteq V(K)$ with $|S|=i$, there is a $j\in J$ with $S\subseteq \bar{B}_j$. Then the graph $G:=K\cup\bigcup_{j\in J}\bar{H}_j$ is n-connected by (Z). It is C_i-critical, as well, because for every complete subgraph $K_{i'}$ of G with $i'\leq i$, there is a vertex $z\in\bigcup_{j\in J}\bar{A}_j$ such that $d(z;G)=n$ and $N(z;G)\supseteq V(K_{i'})$.

Theorem 8. *A C-critical, minimally n-connected graph G cannot contain a complete subgraph K_{n-2}.*

Proof. We suppose that G does contain a K_{n-2}. Let K be a clique containing this K_{n-2}. There is a $T \in \mathfrak{T}(G)$ containing K. By Theorem 7, $n \geq 8$. Hence there is a $b \in V(K)$ with $d(b;G) = n$ by [8]. As $|N(b;G) - T| \leq 3$, there is a $C \in \mathfrak{C}(G-T)$ with $|C \cap N(b;G)| = 1$, say, $C \cap N(b;G) = \{a'\}$. Let K' be a clique containing $\{a'\} \cup N(a';G) \cap K$. In particular, $b \in K'$. There is a $T' \in \mathfrak{T}(G)$ containing K'. As $|C| \geq 2$ and $|\bar{C}| \geq 2$, we get $|C \cap T'| \geq 2$, $|\bar{C} \cap T'| \geq 2$, and $|C' \cap T| \geq 2$ for all $C' \in \mathfrak{C}(G-T')$ from Lemma 10 (2). There is a $C' \in \mathfrak{C}(G-T')$ with $C' \cap K = \emptyset$. Then $C' \cap T = T - K$ and $|K| = n-2$, as $|C' \cap T| \geq 2$. This implies $C' \cap C = \emptyset$ by Lemma 9, as $|\bar{C} \cap T'| \geq 2$. As $C' \cap T = T - K$ and $K' \subseteq T'$, of course, $\bar{C}' \cap T \subseteq K - K'$. As $N(a';G) \cap (K - K') = \emptyset$ by choice of K' and $a' \in C \cap T'$, we see $\bar{C}' \cap C \neq \emptyset$. Then $|C \cap T'| > |C' \cap T|$ by Lemma 9 and so $C' \cap \bar{C} = \emptyset$ by Lemma 1(a). This implies $C' \subseteq T$ and hence $C' = T - K$. As $b \in T'$, there is an $a \in N(b;G) \cap C' \subseteq T - K$. Setting $N(b;G) - K = \{a, a', a''\}$, it follows that $N(b;G) \cap \bar{C} = \{a''\}$. Starting from $\bar{C}$, a'' instead of C, a' and defining K'', T'' analogously, we see as above that $T - K$ is a component of $G - T''$, as well. But then $V(T') = N(T-K;G) = V(T'')$ and hence $\{a', a''\} \subseteq N(T-K;G)$. Then $N(a;G) \cap \{a', a''\} \neq \emptyset$, say, $[a, a'] \in E(G)$. But then b, a, a' span a triangle D. There are a clique $K^* \supseteq D$ and a $T^* \in \mathfrak{T}(G)$ containing K^*. Then by Lemma 10 (2) again, $|C^* \cap T| \geq 2$ for all $C^* \in \mathfrak{C}(G - T^*)$, contradicting the fact that $|T - (K \cup T^*)| \leq 1$. □

Slater's conjecture [13] that there is no non-complete $(2k-1, k)$-graph is obviously equivalent to the conjecture (S_k): *Every non-complete, k-critical graph is (2k)-connected.* Concluding this paper, we will show that for $k \leq 3$, (S_k) remains true, if we replace 'k-critical' by 'C_k-critical'. (If this were true for all k, of course, there would be no C-critical graph.) For $k = 1$, this is well known (see Corollary 1). For $k = 2$, the assertion follows from Tutte's construction-theorem for 3-connected graphs [15]. For the only contraction-critical graph of connectivity number 3 is the complete graph K_4. (Cf. also the paragraph before Example VI.) We now give a proof for $k = 3$.

Theorem 9. *Every C_3-critical graph is 6-connected.*

Proof. We assume $n := \mu(G) \leq 5$. Without loss of generality, we suppose G minimally n-connected. G cannot be C-critical by Theorem 7, hence there exists a $K_4 \subseteq G$ and by [8] there is a $b \in V(K_4)$ with $d(b;G) = n$. There cannot be a complete graph $\bar{K} \subseteq T_b := G(N(b;G))$ with $|\bar{K}| \leq 2$ such that $T_b - \bar{K}$ is complete, because there would be a $\bar{T} \in \mathfrak{T}(G)$ containing $\bar{K} \cup \{b\}$, which would have to separate the complete graph $T_b - \bar{K}$. As $D := T_b \cap K_4$ is a triangle, this implies

$\mu(G)=5$. Define $\{d_1, d_2, d_3\} := V(D)$ and $\{a_1, a_2\} := V(T_b) - D$. First we prove

(Z) $E(T_b) = E(D)$.

As we have seen above, $[a_1, a_2] \notin E(G)$, so it is enough to show that the assumption $[a_1, d_1] \in E(G)$ leads to a contradiction. By C_3-criticality, there is then a $T \in \mathfrak{T}(G)$ containing $\{b, a_1, d_1\}$. As a_2 and $D - T$ belong to different components of $G - T$, there is a $C \in \mathfrak{C}(G - T)$ such that $C \cap T_b = \{a_2\}$. As $a_1 \in T$ and $[a_1, a_2] \notin E(G)$, $|C| \geq 2$ holds. Let K be a clique of $G(\{a_2, d_2, d_3\})$ containing a_2. Then $|K| \leq 2$, as $\bar{C} \cap \{d_2, d_3\} \neq \emptyset$. By C_3-criticality, there is a $T' \in \mathfrak{T}(G)$ containing $\{b\} \cup K$. As T' separates a_1 and $D - T'$ we see $d_1 \in T'$. As $G(\{b\} \cup K)$ or $G(\{b, d_1\} \cup K)$ is a clique in G, we can apply Lemma 10 (2) and get $|C' \cap T| \geq 2$ for all $C' \in \mathfrak{C}(G - T')$, contradicting $|T' \cap T| \geq |\{b, d_1\}| = 2$ and $\mu(G) = 5$.

The graph $G(\{b, a_i\})$ is a clique of G for $i = 1, 2$ by (Z). There is a $T_i \in \mathfrak{T}(G)$ containing $\{b, a_i\}$ for $i = 1, 2$, and $T_b - \{a_1, a_2\}$ being complete, there is a $C_i \in \mathfrak{C}(G - T_i)$ with $C_i \cap T_b = \{a_{i+1}\}$ for $i \in \mathbb{Z}_2$. As T_i contains a clique, $|C_i| \geq 2$ for $i = 1, 2$. So we can apply Lemma 10 (2) and get $|C \cap T_i| \geq 2$ for all $C \in \mathfrak{C}(G - T_{i+1})$ and $i \in \mathbb{Z}_2$. This implies $|C \cap T_i| = 2$ for all $C \in \mathfrak{C}(G - T_{i+1})$ and $T_1 \cap T_2 = \{b\}$. Hence $C_i \subseteq T_{i+1}$ for $i \in \mathbb{Z}_2$ by Lemma 9 and so $|C_i| = 2$ for $i = 1, 2$. Since $C_1 \cap T_b = \{a_2\}$, $\bar{D} := G(C_1 \cup C_2) - \{a_2\}$ is a triangle. Hence there is a $\bar{T} \in \mathfrak{T}(G)$ containing $\bar{D}$. Then $C_1 \cup C_2 \subseteq \bar{T}$ by Lemma 8, since $[a_1, a_2] \notin E(G)$. Since $T_1 \cap T_2 = \{b\}$, there is no vertex z with $N(z; G) \supseteq V(C_1 \cup C_2)$. So Lemma 4 implies $|\bar{C} \cap T_1| \geq 2$ for all $\bar{C} \in \mathfrak{C}(G - \bar{T})$, which is incompatible with $|\bar{T} \cap T_1| \geq |C_2| = 2$. $\square$

Finally, we will touch briefly upon the infinite case. First, we shall show by an example that the results of this paper are not true for infinite graphs, in general.

Example VII. Let T be a tree regular of degree 4. Assign a quadrangle $Q(x)$ to every $x \in V(T)$ such that $Q(x) \cap Q(y) = \emptyset$ for $x \neq y$. For every $[x, y] \in E(T)$, identify an edge of $Q(x)$ with an edge of $Q(y)$ in such a way that every edge of any quadrangle is identified with exactly one other edge. This is obviously possible. The resulting graph has connectivity number 2 and is contraction-critical, but it has no triangles. Hence it is even C-critical and represents a counterexample to the conclusions of Theorems 7 and 9 in the infinite case.

The graph constructed above is regular of degree $\aleph_0$. As usual, for locally finite graphs, we have a better chance to get results similar to the results in the finite case.

Theorem 10. *Every contraction-critical, locally finite, infinite graph G has an infinite number of triangles.*

Sketch of Proof. Choose any $z \in V(G)$ and define $\mathfrak{S} := \{V(e): e \in E(G)$ and $z \in V(e)\}$. First we prove

(Z) *There is a finite $\mathfrak{S}$-fragment of G.*

Consider an $\mathfrak{S}$-fragment F of G such that $|F \cap N(z; G)|$ is minimal. There are an $x \in F \cap N(z; G)$ and a $T \in \mathfrak{T}(G)$ containing $\{x, z\}$. Of course, T is finite, since G is locally finite. So we may assume that there is a $C \in \mathfrak{C}(G - T)$ with $C \cap F \neq \emptyset$. As $|N(z; G) \cap C \cap F| < |N(z; G) \cap F|$, $C \cap F$ is not an $\mathfrak{S}$-fragment and so $\bar{C} \cap \bar{F} = \emptyset$ as in Lemma 1(b). If $\bar{C} \cap F \neq \emptyset$, then also $C \cap \bar{F} = \emptyset$, and hence $\bar{F} \subseteq T$ is a finite $\mathfrak{S}$-fragment. If $\bar{C} \cap F = \emptyset$, then $\bar{C} \subseteq T_F$ is a finite $\mathfrak{S}$-fragment.

By (Z) an $\mathfrak{S}$-atom A of G is finite and hence Theorem 1 remains true for A, as is easily seen. But then Theorem 3 follows in the same way as in the finite case. Theorem 3 obviously implies that there are an infinite number of triangles in G.

It is no problem to check that also Theorem 5 remains true for locally finite graphs, whereas the proof of Theorem 8 works even for all graphs of finite connectivity number $n \geqslant 5$.

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ON THE EUCLIDEAN DIMENSION OF A COMPLETE MULTIPARTITE GRAPH

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The euclidean dimension of a graph G, $e(G)$, is the minimum n such that the vertices of G can be placed in euclidean n-space, R^n, in such a way that adjacent vertices have distance 1 and nonadjacent vertices have distances other than 1. Let $G = K(n_1, \ldots, n_{s+t+u})$ be a complete $(s+t+u)$-partite graph with vertex-classes consisting of s sets of size 1, t sets of size 2, and u sets of size $\geqslant 3$. We prove that $e(G) = s+t+2u$ if $t+u \geqslant 2$, and $e(G) = s+t+2u-1$ if $t+u \leqslant 1$.

1. Introduction

Let U^n be the *unit distance graph* of euclidean n-space, R^n, i.e. the vertex set of U^n is R^n and two vertices are adjacent if and only if the distance between them is 1. For a simple graph G, the *euclidean dimension* of G, $e(G)$, is the minimum integer n such that G is isomorphic to an *induced subgraph* of U^n.

Erdös et al. [2] defined the dimension of a graph G, $\dim(G)$, as the minimum n such that G is isomorphic to a *subgraph* of U^n. Recently, Buckley and Harary [1] introduced the term "euclidean dimension" in the present sense.

The euclidean dimension of the complete graph K_p is clearly $p-1$. The euclidean dimension of the complete bipartite graph $K(p, q)$ was obtained by Lenz (see [2]).

Theorem A (Lenz). *The euclidean dimensions of the complete bipartite graphs are as follows*:

$$e(K(1,1)) = e(K(1,2)) = 1; \qquad e(K(1,n)) = 2 \quad \textit{for} \quad n > 2;$$
$$e(K(2,2)) = 2; \qquad e(K(2,n)) = 3 \quad \textit{for} \quad n > 2;$$
$$e(K(m,n)) = 4 \quad \textit{for} \quad m, n \geqslant 3.$$

Buckley and Harary [1] determined the euclidean dimension of the complete tripartite graph $K(p, q, r)$.

Theorem B (Buckley and Harary). *The euclidean dimension of complete tripartite*

graph $K(p, q, r)$, $p \leq q \leq r$, *is given by*

$$e(K(p,q,r)) = \begin{cases} 2 & \text{if } p = q = 1, \quad r \leq 2 \\ 3 & \text{if } p = q = 1, \quad r \geq 3 \text{ or } q = r = 2 \\ 4 & \text{if } q = 2, \quad r \geq 3 \\ 5 & \text{if } p \leq 2, \quad q \geq 3 \\ 6 & \text{if } p \geq 3. \end{cases}$$

In this paper we present the following general result.

Theorem C. *Let*

$$G(s, t, u) = K(\underbrace{1, \ldots, 1}_{s}, \underbrace{2, \ldots, 2}_{t}, n_1, \ldots, n_u)$$

be a complete $(s+t+u)$*-partite graph with* $n_i \geq 3$, s, t, u *being possibly* 0, $s+t+u \geq 2$. *Then*

$$e(G(s,t,u)) = \begin{cases} s+t+2u & \text{if } t+u \geq 2 \\ s+t+2u-1 & \text{if } t+u \leq 1. \end{cases}$$

2. Proof

For a nonempty set X in R^n, let $L(X)$ denote the affine subspace spanned by X. First we prove the following lemma.

Lemma 1. *Let* X, Y *be two nonempty sets in* R^n *such that each point of* X *is unit distance apart from all points of* Y. *Then*

$$\dim L(X \cup Y) = \dim L(X) + \dim L(Y) + \varepsilon,$$

where $\varepsilon = 0$ *if* $L(X) \cap L(Y) \neq \emptyset$ *and* $\varepsilon = 1$ *otherwise. Furthermore, if* $L(X)$ *and* $L(Y)$ *intersect, then they intersect perpendicularly at the common center of the two spheres, the minimum-radius sphere passing through all the points of* X, *and the minimum-radius sphere passing through all the points of* Y.

Proof. If X is a singleton set then the lemma follows immediately. Suppose that X contains at least two points. Take a point y of Y and let z be the orthogonal projection of y on $L(X)$. Since

$$\|y - z\|^2 + \|z - x\|^2 = 1 \quad \text{for all } x \text{ of } X,$$

z is the center of a sphere that passes through all points of X, and since z is on $L(X)$, this sphere must be the minimum-radius sphere that passes through all the points of X. Therefore the point z is independent of the choice of y in Y. Thus

the vectors $\overrightarrow{zx}$ and $\overrightarrow{zy}$ $(x \in X, y \in Y)$ are always orthogonal, and we have

$$\dim L(X \cup Y) = \dim L(X) + \dim L(Y \cup \{z\}),$$
$$L(X) \cap L(Y) \subset L(X) \cap L(Y \cup \{z\}) = \{z\}.$$

Now, if $L(X) \cap L(Y) \neq \emptyset$, then $z \in L(Y)$ i.e. $\dim L(Y \cup \{z\}) = \dim L(Y)$, and in this case z must also be the center of the minimum-radius sphere that passes through all the points of Y. If $L(X) \cap L(Y) = \emptyset$, then z is not in $L(Y)$ and $\dim L(Y \cup \{z\}) = \dim L(Y) + 1$. □

The following lemma is easy, and its proof is omitted (see e.g. [3]).

Lemma 2. *Let $w_1, \ldots, w_s$ be the vertices of a regular simplex in R^n with side length* 1, *and z be the 'center' of the simplex. Then*

$$\|w_i - z\|^2 = \frac{(s-1)}{(2s)}.$$

The complement of a graph G is denoted by $\bar{G}$. The join $G + H$ of two graphs G and H is the complement of the disjoint union of $\bar{G}$ and $\bar{H}$. For an integer $p \geq 2$, we define

$$f(p) = \begin{cases} 1 & \text{if } p = 2 \\ 2 & \text{if } p \geq 3. \end{cases}$$

Lemma 3. *For $p, q \geq 2$, the following hold:*

(1) $e(G + \bar{K}_p) \geq e(G) + f(p)$,
(2) $e(K(p, q)) = f(p) + f(q)$,
(3) $e(K_s + \bar{K}_p) = s - 1 + f(p)$,
(4) $e(K_s + K(p, q)) = s + f(p) + f(q)$.

Proof. (1) Let $m = e(G + \bar{K}_p)$. Embed $G + \bar{K}_p$ in U^m as an induced subgraph and let X, Y be the vertex sets of G and $\bar{K}_p$ in this embedding, respectively. Then since Y is spherical (i.e. all points of Y lie on a sphere), we have that $\dim L(Y) \geq f(p)$. Hence by Lemma 1, we have

$$m \geq \dim L(X) + \dim L(Y) \geq e(G) + f(p).$$

(2) Similarly, we have $e(K(p, q)) = e(\bar{K}_p + \bar{K}_q) \geq f(p) + f(q)$. To prove the opposite inequality, take p points $x_1, \ldots, x_p$ in $R^{f(p)}$ in such a way that $\|x_i\|^2 = \frac{1}{2}$ and $\|x_i - x_j\| \neq 1$ for every i, j. This is clearly possible. Similarly, take q points $y_1, \ldots, y_q$ in $R^{f(q)}$ so that $\|y_i\|^2 = \frac{1}{2}$ and $\|y_i - y_j\| \neq 1$. Then, the $p + q$ points

$$(x_i, 0) \quad i = 1, \ldots, p \quad \text{and} \quad (0, y_j) \quad j = 1, \ldots, q$$

in $R^{f(p)+f(q)} = R^{f(p)} \times R^{f(q)}$ together induce a graph isomorphic to $K(p, q)$. Hence, $e(K(p, q)) \leq f(p) + f(q)$.

(3) by (1), $e(K_s+\bar{K}_p) \geq e(K_s)+f(p)=s-1+f(p)$. Let $w_1,\ldots,w_s$ be the vertices of a regular simplex of side length 1 in R^{s-1} which is centered at the origin. Then $\|w_i\|^2=(s-1)/(2s)$. Take p points $x_1,\ldots,x_p$ in $R^{f(p)}$ so that $\|x_i\|^2=(s+1)/(2s)$ and $\|x_i-x_j\| \neq 1$. Then the points

$$(w_i, 0) \quad i=1,\ldots,s \quad \text{and} \quad (0, x_j) \quad j=1,\ldots,p$$

in $R^{s-1+f(p)}=R^{s-1}\times R^{f(p)}$ induce $K_s+\bar{K}_p$.

(4) Let $n=e(K_s+K(p,q))$. First we show that $n \geq s+f(p)+f(q)$. Embed $K_s+K(p,q)=K_s+\bar{K}_p+\bar{K}_q$ in U^n as an induced subgraph, and let W, X, Y be the vertex sets of K_s, $\bar{K}_p$, and $\bar{K}_q$, respectively. Then $\dim L(W)=s-1$, $\dim L(X) \geq f(p)$, and $\dim L(Y) \geq f(q)$. Suppose that any two of $L(W)$, $L(X)$, $L(Y)$ has a common point. Then by Lemma 1, these three affine subspaces must have a common point z, which is the common center of the three spheres passing through the points of W, the points of X, and the points of Y. Since $\|w-z\|^2=(s-1)/(2s)$ for all w of W by Lemma 2, we must have $\|x-z\|^2=\|y-z\|^2=(s+1)/(2s)$ for x of X and y of Y. Hence, we have $\|x-y\|^2=\|x-z\|^2+\|y-z\|^2>1$, a contradiction. Thus, there must be a disjoint pair in $L(W)$, $L(X)$, $L(Y)$. Then, again by Lemma 1, we can see that $n \geq \dim L(W)+\dim L(X)+\dim L(Y)+1 \geq s+f(p)+f(q)$.

To show the opposite inequality, let $w_1,\ldots,w_s$ be the vertices of a regular simplex of side length 1 in R^{s-1} centered at the origin, and let $\bar{w}_i=(w_i, 1/\sqrt{2s})$ in $R^{s-1}\times R=R^s$. Take x_i and y_j as in the proof of (2). Then the $s+p+q$ points

$$(\bar{w}_i,0,0)\ i=1,\ldots,s, \qquad (0,x_j,0)\ j=1,\ldots,p, \qquad (0,0,y_k)\ k=1,\ldots,q$$

in $R^s\times R^{f(p)}xR^{f(q)}=R^{s+f(p)+f(q)}$ induce $K_s+K(p,q)$. Hence $n \leq s+f(p)+f(q)$. □

Proof of Theorem C. If $t+u \leq 1$, then by (3) of Lemma 3, we have $e(G(s,t,u))=s-1+t+2u$. Now we prove that $e(G(s,t,u))=s+t+2u$ for $t+u \geq 2$ and furthermore, that there is a point set on a sphere of radius $1/\sqrt{2}$ in R^{s+t+2u}, which induces a subgraph isomorphic to $G(s,t,u)$. The proof is by induction on $t+u$. The case $t+u=2$ follows from (4) (and its proof) of Lemma 3. Suppose that $e(G(s,t,u))=s+t+2u$ and let X be a point set on the sphere of radius $1/\sqrt{2}$ in R^{s+t+2u} centered at the origin. We need to show that $e(G(s,t,u)+\bar{K}_p)=s+t+2u+f(p)$, $p \geq 2$, and that there is a point set of a sphere of radius $1/\sqrt{2}$ in $R^{s+t+2u+f(p)}$, which induces a subgraph isomorphic to $G(s,t,u)+\bar{K}_p$. Let Y be a point set on the sphere of radius $1/\sqrt{2}$ in $R^{f(p)}$ centered at the origin, which induces a subgraph isomorphic to $\bar{K}_p$. Then the points

$$(x,0), \qquad (0,y)\in R^{s+t+2u}\times R^{f(p)} \quad (x\in X, y\in Y)$$

are clearly on a sphere of radius $1/\sqrt{2}$, and they induce together a subgraph isomorphic to $G(s,t,u)+\bar{K}_p$. Hence $e(G(s,t,u)+\bar{K}_p) \leq e(G(s,t,u))+f(p)$. The opposite inequality follows from (1) of Lemma 3. □

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COMPUTATION OF SOME CAYLEY DIAGRAMS

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Reidemeister–Schreier method gives generators and relations of a subgroup of finite index of a group defined by generators and relations. In this note we introduce a graph-theoretical explanation of this method and we show two examples of its application. Let $G = \langle a, b \mid a^3, b^2, R\rangle$, that is, G is generated by elements a of order 3 and b of order 2 with one relation $R = 1$. We show that G with R of length at most 22 whose order cannot be computed by coset enumeration in [6] is of infinite order. Next let $G = \langle a, b \mid a^l, b^m, R\rangle$ with $R = W^n$ and $l, m, n > 1$. Baumslag et al. [1] obtained that G has a representation where the images of a, b and W are of order l, m and n respectively and that G is infinite if $1/l + 1/m + 1/n \leq 1$. We prove the latter part of these results by using Reidemeister–Schreier method.

Reidemeister–Schreier method

Let G be a group which is generated by a subset S of G and let H be a subgroup of G. Let Γ be a directed graph with vertex set G/H and arc set

$$\{(Hx, Hxs, s) \mid x \in G, s \in S\}.$$

We consider (Hx, Hxs, s) to be an arc from Hx to Hxs which is labeled by s and at the same time to be an arc from Hxs to Hx which is labeled by s^{-1}, Γ is called the *Cayley diagram* or the *Schreier diagram* of G/H with respect to S according as H is the identity group or not. If H is normal in G, we also call Γ the *Cayley diagram* of the factor group G/H.

Now for a spanning tree of Γ we assign a distinct new letter to each of the arcs that are not contained in the tree, and also give an orientation arbitrary fixed. In Reidemeister–Schreier method these new letters are the set of generators of H (see [5]). Suppose that relations of G with the elements of S as generators are given. Then the relations of H are the collection of the following sequences: we trace each relation of G from each vertex along the label and we have a sequence of new letters and their inverses according to the directions tracing along or against the orientations.

Tracing each relation gives a cycle (a closed walk) in Γ, and the arcs assigned new letters form a basis of the cycle space of Γ, considering each of them together with the spanning tree. If we assume all the generators of H commute mutually, then we have a presentation of the maximal abelian factor group of H. Graph-theoretically the cycle space generated by the relations gives the structure

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of the factor group. In particular its order is infinite if and only if the rank of the cycle space is less than that of Γ.

Application

Let $G = \langle a, b \mid a^3, b^2, R\rangle$. The cyclic conjugates or the inverse of a relation defines the same group, and so does interchanging a and a^{-1}. So we do not distinguish these relations. The length of R is defined by the total number of $a^{\pm 1}$ and b in the sequence $a^{\pm 1}ba^{\pm 1}b \cdots a^{\pm 1}b$ of the relation R. Matsuyama [6] tried to compute the orders of all groups with R of length at most 30 by coset enumeration program. Following is the list of the relations R of length at most 22 that cannot be computed in [6].

$$(ab)^n, \qquad n = 6, 7, \ldots, 11,$$
$$(aba^{-1}b)^n, \qquad n = 3, 4, 5,$$
$$((ab)^4a^{-1}b)^2 = (ab)^{-1}(ab)^6(b(ab)^6b)ab,$$
$$(ab)^3a^{-1}bab(a^{-1}b)^3aba^{-1}b = (aba^{-1}b)^3(bab)^{-1}(aba^{-1}b)^3bab,$$
$$((ab)^3(a^{-1}b)^2)^2,$$
$$(ab)^2a^{-1}b(ab)^2(a^{-1}b)^2ab(a^{-1}b)^2.$$

All but the last two cases are known to be infinite or have factor groups which are known to be infinite (see [3, 7]).

Case $((ab)^3(a^{-1}b)^2)^2$. First we note that it is easily seen that generally in Reidemeister–Schreier method we do not need to trace a relation $b^2 = 1$ on a diagram by only identifying the two opposite arcs (Hx, Hxb, b) and (Hxb, Hx, b) if the two vertices Hx and Hxb are different from each other. Then for the case, adding a relation $(ab)^4 = 1$, we have a truncated hexahedron as the diagram by coset enumeration. If we take a spanning tree which contains two arcs of every triangle, then the relation a^3 gives trivial generators. So the rank of the cycle space is easily computed and we have this case of infinite order.

Case $(ab)^2a^{-1}b(ab)^2(a^{-1}b)^2ab(a^{-1}b)$. Adding a relation $(ab)^3 = 1$ gives a truncated tetrahedron and we have that the order is infinite as above.

Next let $G = \langle a, b \mid a^l, b^m, W^n\rangle$. Suppose that the image $\bar{G}$ of G by the representation obtained in [1] is of finite order g. Let H be the kernel of this representation. Then in the Cayley diagram Γ of $\bar{G}$ which is a 2-regular directed graph without loops, clearly there exist $g + 1$ arcs not belonging to a spanning tree. Since the image of a is of order l and since a Cayley diagram is afforded by a regular representation, we have g/l cycles by tracing the relation a^l in Γ (cf. [4]).

Similarly we have g/m and g/n cycles by tracing b^m and W^n respectively, and in total we have $g/l+g/m+g/n$ cycles. Hence H has $g+1$ generators and $g/l+g/m+g/n$ relations. So if $1/l+1/m+1/n \leq 1$, then the number of the relations of H is less than that of the generators of H. This implies that G is infinite.

Note added in proof

Conder [2] determined all the groups $\langle a, b \mid a^3, b^2, R\rangle$ with R of length at most 24 in detail.

Except three relations the author is now able to show the finiteness and infiniteness of the groups with R of length at most 30.

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SHORT CYCLES IN DIGRAPHS

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Caccetta and Häggkvist [1] conjectured that every digraph with n vertices and minimum outdegree k contains a directed cycle of length at most $\lceil n/k \rceil$. With regard to this conjecture, Chvátal and Szemerédi [2] proved that if G is a digraph with n vertices and if each of these vertices has outdegree at least k, then G contains a cycle of length at most $(n/k)+2500$.
Our result is an improvement of this result.

Main theorem. *Let G be a digraph with n vertices such that each vertex of G has outdegree at least k ($\geqslant 2$). Then G has a cycle of length at most $(n/k)+304$.*

Proof. Our proof is a straightforward refinement of the highly original argument of [2]. So we shall simply sketch the proof. We proceed by induction on n. For convenience, we write $t=n/k$ and denote the number of edges in a shortest directed path from x to y by dist(x, y). As in [2], we may assume that $t \geqslant 304$, and that there is a vertex x such that at most $\frac{1}{2}(n+1)$ vertices y have dist$(x, y) \leqslant \lfloor \frac{1}{2}t \rfloor + 152$.

First, we define c using the equation $(\lfloor \frac{1}{2}t \rfloor + 152)(k-c) = \frac{1}{2}(n+1)$, and consider the smallest positive integer d with the following property:

there is a vertex s such that at most

$$d(k-c) \text{ vertices } y \text{ have } \operatorname{dist}(s, y) \leqslant d. \quad (1)$$

With s as in (1), we can show that, for all nonnegative integers i,

$$\text{at most } (i+1)(k-c) \text{ vertices } y \text{ have } d-i \leqslant \operatorname{dist}(s, y) \leqslant d. \quad (2)$$

For later reference, note the following inequalities:

$$\tfrac{1}{2}k > c > \frac{(302k-1)}{(t+304)} \geqslant \frac{(150.75)k}{t}. \quad (3)$$

Case 1. For every vertex v with dist$(s, v) = d-1$, there is a vertex w with dist$(v, w) < (t/74.3) - 1$ and dist$(s, w) \leqslant d-3$.

In this case, we consider the following two subsets of the vertex set of G.

$$Q := \{v \mid \mathrm{dist}(s, v) = d - 1\}$$

$$P := \{u \mid \mathrm{dist}(s, u) = d - 2 \text{ and there are at least } k/4.2 \text{ edges } uv \text{ with } v \in Q\}$$

First we shall show that there is a subset R of P and a subset S of Q such that $|R| \leqslant (7.4/8.4)^{19} k$, $|S| \leqslant 19$ and such that for every u in $P - R$ there is an edge uv with $v \in S$. In proving this assertion, we may assume that $|Q| > 19$, for otherwise we could set $R = \emptyset$ and $S = Q$.

By (2) with $i = 1$ and by (3), we have $|Q| \leqslant 2k$. Since P and Q are disjoint, we similarly get $|P| + |Q| = |P \cup Q| \leqslant 3k$ by (2) with $i = 2$. Setting $|Q| = \varepsilon k$ $(0 \leqslant \varepsilon \leqslant 2)$, we have $|P| \leqslant (3 - \varepsilon)k$. Note that if $P = \emptyset$, we can simply set $R = S = \emptyset$. Therefore we may assume $P \neq \emptyset$. Thus we have $|Q| \geqslant k/4.2$ by the definition of P, and so $\varepsilon \geqslant 1/4.2$.

We can now construct as in [2] a sequence $R_0, \ldots, R_{19}$ of subsets of P and a sequence $S_0, \ldots, S_{19}$ of subsets of Q such that $|R_i| \leqslant (3 - \varepsilon)(1 - 1/(4.2\varepsilon))^i k$, $|S_i| = i$ and such that R_i consists of those vertices u in P for which there is no edge uv with $v \in S_i$.

Since $\max_{1/4.2 \leqslant \varepsilon \leqslant 2} (3 - \varepsilon)(1 - 1/(4.2\varepsilon))^{19} = (7.4/8.4)^9$, we may set $R = R_{19}$ and $S = S_{19}$.

Next, we consider the following subset of $V(G)$:

$$T := \{y \mid \mathrm{dist}(s, y) \leqslant d - 2 \text{ and } y \notin R\}.$$

We shall construct a graph H by adding certain new edges to the subgraph of G induced by T. These new edges uz are added for each vertex u in $P - R$, while z runs through all the vertices for which there is an edge uv with $v \in S$ and $\mathrm{dist}(v, z) < t/74.3$. Since $d \leqslant \lfloor \frac{1}{2}t \rfloor + 152$, at most $\frac{1}{2}(n + 1)$ vertices y in G have $\mathrm{dist}(s, y) \leqslant d$. If $Q = \emptyset$, there are no edges directed from Z to $V(G) - Z$, where

$$Z := \{y \mid \mathrm{dist}(s, y) \leqslant d - 2\}.$$

We get the desired conclusion immediately, by applying the induction hypothesis to the subgraph induced by Z. Thus we may assume $Q \neq \emptyset$, and therefore, we have $|V(H)| = |T| < \frac{1}{2}n$.

By estimating the outdegree of the vertices in H (following closely the argument in [2]), we get

$$\deg_H^+(u) \geqslant (1 - 1/4.2 - (7.4/8.4)^{19})k$$

for all $u \in T$, where $\deg_H^+(u)$ denotes the outdegree of u in H. We include here the proof for the case $u \in P - R$. If $u \in P - R$, then there is an edge uv with $v \in S$. By the assumption of Case 1, there is a vertex w with $\mathrm{dist}(v, w) < (t/74.3) - 1$ and $\mathrm{dist}(s, w) \leqslant d - 3$. Then there are at least $k - (7.4/8.4)^{19}k$ vertices z in T with $wz \in E(G)$. (*Note*: We may assume that u is not among these vertices z. For if so, it means that G contains a directed cycle of length at most $(t/74.3) + 1$.) Hence $\deg_H^+(u) \geqslant (1 - (7.4/8.4)^{19})k \geqslant (1 - 1/4.2 - (7.4/8.4)^{19})k$. Since $0.5/(1 - 1/4.2 -$

$(7.4/8.4)^{19}) + 19/74.3 < 1$, the rest of the proof corresponds word for word to that in [2].

Case 2. There is a vertex v with $\operatorname{dist}(s, v) = d - 1$ such that no vertex w has $\operatorname{dist}(v, w) < (t/74.3) - 1$ and $\operatorname{dist}(s, w) \leqslant d - 3$.

In this case, we consider the following subset of $V(G)$,

$$R_j := \{y \mid \operatorname{dist}(v, y) \leqslant j \text{ and } d - 2 \leqslant \operatorname{dist}(s, y) \leqslant d - 1\},$$
$$U := \{w \mid \operatorname{dist}(s, w) = d\}.$$

By an argument similar to the one in Case 1, we get

$$|U| + |R_j| \leqslant 3(k - c) \qquad (U \cap R_j = \emptyset).$$

Hence if we set $|U| = \varepsilon(k - c)$ $(0 \leqslant \varepsilon \leqslant 1)$, then we have $|R_j| \leqslant (3 - \varepsilon)(k - c)$ for all j. On the other hand, for all sufficiently large j,

$$|R_j| \leqslant (2 - \varepsilon)(74.3/75.3)jc. \tag{4}$$

Consider the smallest j satisfying (4); we claim that

$$j < (t/74.3) - 1. \tag{5}$$

By the minimality of j, we have

$$(2 - \varepsilon)(74.3/75.3)(j - 1)c < |R_{j-1}| \leqslant (3 - \varepsilon)(k - c).$$

This, together with (3), implies

$$\begin{aligned} j - 1 &< (75.3/74.3)((3 - \varepsilon)/(2 - \varepsilon))(t/150.75 - 1) \\ &\leqslant 2(75.3/74.3)(t/150.75 - 1), \end{aligned}$$

and (5) follows as $t \geqslant 304$.

We consider the subgraph F induced by R_{j-1}. By estimating the outdegree of the vertices y in F (following closely the argument in [2]), we have

$$\deg_F^+(y) \geqslant k - \varepsilon(k - c) - (2 - \varepsilon)(74.3/75.3)c.$$

Now (3) implies

$$\deg_F^+(y) > (2 - \varepsilon)c/75.3.$$

By the induction hypothesis, F contains a directed cycle of length at most

$$\frac{(2 - \varepsilon)(74.3/75.3)jc}{(2 - \varepsilon)c/75.3} + 304 = 74.3j + 304 < t + 304.$$

This completes the proof. $\square$

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SUBGRAPH COUNTS IN RANDOM GRAPHS USING INCOMPLETE *U*-STATISTICS METHODS

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The random graph $K_{n,p}$ is constructed on n labelled vertices by inserting each of the $\binom{n}{2}$ possible edges independently with probability p, $0 > p < 1$. For a fixed graph G, the threshold function for existence of a subgraph of $K_{n,p}$ isomorphic to G has been determined by Erdös and Rényi [8] and Bollobás [3]. Bollobás [3] and Karoński [14] have established asymptotic Poisson and normal convergence for the number of subgraphs of $K_{n,p}$ isomorphic to G for sequences of $p(n) \to 0$ which are slightly greater than the threshold function. We use techniques from asymptotic theory in statistics, designed to study sums of dependent random variables known as U-statistics. We note that a subgraph count has the form of an incomplete U-statistic, and prove asymptotic normality of subgraph counts for a wide range of values of p, including any constant p and sequences of $p(n)$ tending to 0 or 1 sufficiently slowly.

1. Introduction

Erdös and Rényi [8] introduced the random graph model $K_{n,p}$, in which a graph is constructed on n labelled vertices with each of the $\binom{n}{2}$ possible edges present with probability p, $0 < p < 1$, independently. Without loss of generality, we will label the vertices $\{1, 2, \ldots, n\}$. We may view the random graph $K_{n,p}$ as determined by a set $\{X(i, j)\}$, $1 \leq i < j \leq n$, of independent Bernoulli random variables with $p = \{X(i, j) = 1\} = 1 - P(X(i, j) = 0)$ for all $1 \leq j < j \leq n$, where $X(i, j) = 1$ indicates that an edge is present between i and j, and $X(i, j) = 0$ indicates the absence of an edge between i and j. For convenience, we let $q = 1 - p$.

One often views a random graph as a structure which evolves as edges are added successively, or as p increases from 0 to 1. Many important properties of graphs appear suddenly in this evolution as the probability p crosses a threshold, on opposite sides of which $K_{n,p}$ has the property with probability 0 or 1 asymptotically as $n \to \infty$. For most properties of interest, the threshold is a function of n which tends to 0 as $n \to \infty$. For a given function $p(n)$, a property is said to hold for *almost all graphs* if it holds for a set of random graphs $K_{n,p}$ with

probability tending to one as $n \to \infty$. A more complete discussion of the extensive literature on random graphs is available in the recent introduction by Palmer [17] and research monograph by Bollobás [4].

For a fixed graph G, we consider the *subgraph count*, $S_n(G)$, a random variable defined as the number of subgraphs of $K_{n,p}$ which are isomorphic to G. Introduce the indicator function notations

$$I(A \subseteq K_{n,p}) = \begin{cases} 1 & \text{if } A \text{ is a subgraph of } K_{n,p} \\ 0 & \text{otherwise} \end{cases}$$
$$= \prod_{e \in E(A)} X(e)$$

where $E(A)$ denotes the edge set of graph A.

$$I(A \sim G) = \begin{cases} 1 & \text{if } A \text{ is isomorphic to } G \\ 0 & \text{otherwise} \end{cases}$$

We may express the subgraph count as

$$S_n(G) = \sum_{A \subseteq K_n} I(A \sim G) I(A \subseteq K_{n,p})$$
$$= \sum_{A \subseteq K_n} I(A \sim G) \prod_{e \in E(A)} X(e)$$

where K_n denotes the complete graph on the vertex set $\{1, 2, \ldots, n\}$. Note that we identify a graph with its set of edges, so the approach applies directly only to graphs G with no isolated vertices. However, the results can be easily extended to graphs which have isolated vertices.

If G is a graph with k edges, we need only consider subgraphs A with k edges, so

$$S_n(G) = \sum_{\substack{A \subseteq K_n \\ |E(A)| = k}} I(A \sim G) \prod_{e \in E(A)} X(e)$$

which has the form

$$\sum_{\substack{e_1, e_2, \ldots, e_k \text{ distinct} \\ e_i \in E(K_n)}} w(e_1, e_2, \ldots, e_k) h(X(e_1), X(e_2), \ldots, X(e_k)) \tag{$*$}$$

in which $w(e_1, \ldots, e_k)$ is a nonrandom indicator function.

The primary goal of this paper is to establish that the subgraph count random variable $S_n(G)$ has an asymptotic normal distribution for all graphs G with no isolated vertices, and for a broad range of probability functions $p(n)$. A secondary purpose is to introduce statistical methods for the treatment of random variables of the form (*), which are called weighted U-statistics, which may be relevant to other random graph problems. Previous results on the threshold for existence of a subgraph of $K_{n,p}$ isomorphic to G, and on limiting Poisson and normal distributions for subgraph counts for sequences $p(n)$ converging to 0 but slightly above the existence threshold, are discussed in Section 2. The principal

result, stated in Section 3, establishes asymptotic normality of subgraph counts when p is constant, and when the sequence $p(n)$ converges to 0 or 1 sufficiently slowly. The result fills a substantial gap in the literature on asymptotic subgraph count distributions. Examples illustrating the application of our result to stars, trees, cycles, and complete graphs are also presented in Section 3. Section 4 discusses statistical tools for treating sums of dependent random variables, from the theory of U-statistics, and adapts them for treatment of subgraph counts. For computation of variances of the relevant statistics, counts of intersections of isomorphic subgraphs are considered in Section 5.

2. Previous subgraph count results

The determination of the threshold for the existence of a given graph G as a subgraph of $K_{n,p}$ was the focus of Theorem 1 of Erdös and Rényi [8]. For this problem, Erdös and Rényi introduced the concept of a balanced graph. Define the *degree* of a graph G by

$$d(G) = |E(G)|/|V(G)|.$$

A graph G is *balanced* if $d(G) \geq d(H)$ for every subgraph H of G. Let $A \supset B$ denote that A contains a subgraph isomorphic to B. Erdös and Rényi proved that if G is balanced

$$\lim_{n\to\infty} P[K_{n,p} \supset G] = \begin{cases} 0 & \text{if } p(n)n^{1/d(G)} \to 0 \text{ as } n\to\infty \\ 1 & \text{if } p(n)n^{1/d(G)} \to \infty \text{ as } n\to\infty \end{cases}$$

which identifies the threshold function for existence of a balanced graph G as $n\to\infty$.

Bollobás [3] generalized this result to arbitrary graphs G. Define $m(G)$ as the maximal degree of any subgraph of G. Note that $m(G) = d(G)$ if and only if G is balanced. Then for any graph G,

$$\lim_{n\to\infty} P[K_{n,p} \supset G] = \begin{cases} 0 & \text{if } p(n)n^{1/m(G)} \to 0 \text{ as } n\to\infty \\ 1 & \text{if } p(n)n^{1/m(G)} \to \infty \text{ as } n\to\infty \end{cases}$$

i.e. the existence threshold is $n^{-1/m(G)}$. The Bollobás proof uses a rather intricate method called grading, but a short elementary proof has been supplied by Rucinski and Vince [19] using the second moment method.

A graph G is *strictly balanced* if $d(G) > d(H)$ for every proper subgraph H of G. This concept plays a crucial role in obtaining asymptotic distributions for subgraph counts near the threshold function for existence. Independently, Bollobás [3] and Karoński and Rucinski [15] proved the following: Let G be a strictly balanced graph with k edges, l vertices, $m = m(G) = k/l$, and an automorphism group of order a. Let $p(n)n^{1/m} \to c$ as $n\to\infty$, for some $0 < c < \infty$. Then $S_n(G)$ has an asymptotic Poisson distribution with mean c^k/a.

Research of Rucinski and Vince [19] shows that the factorial moment

convergence method for establishing Poisson convergence does not apply if G is not strictly balanced. While their result does not prove that Poisson convergence is impossible, it strongly suggests that Poisson convergence of a subgraph count holds if and only if the graph is strictly balanced.

For a small range of sequences $p(n)$ above the existence threshold, Karoński and Rucinski [15] established asymptotic normality for subgraph counts. If G is a strictly balanced graph, and $p(n)n^{1/m(G)} \to \infty$, but for any $\delta > 0$, $p(n)n^{1/m(G)-\delta} \to 0$, then $S_n(G)$ is asymptotically normally distributed.

Novicki [16] treats multiple subgraph count statistics, obtaining multivariate normal limiting distributions, and also treats induced subgraph count statistics, for constant values of p.

Janson [13] applies the method of semi-invariants to derive limiting normal distributions for induced subgraph counts to deal with graphs for which the usual normalization is not valid for certain constant values of p.

3. Statement of results

Theorem 3.1. *Let G be a graph with no isolated vertices. Suppose that G has k edges and l vertices, and that*

$$np^{k-1} \to +\infty$$

$$n^2(1-p) \to +\infty.$$

Then

$$\frac{S_n(G) - E[S_n(G)]}{\binom{n-2}{l-2}\frac{2k}{a}(l-2)!\sqrt{\binom{n}{2}p(1-p)}}$$

has an asymptotic Normal (0, 1) *distribution.*

Remark 1. Let G' be a graph with $l+m$ vertices, m of which are isolated vertices, and let G be the graph obtained by deleting the isolated vertices from G'. For an isomorphic copy of G on a set of l vertices, there are $\binom{n-l}{m}$ sets of vertices that may be added to obtain a copy of G'. Thus, $S_n(G') = \binom{n-l}{m}S_n(G)$, and the asymptotic distribution for $S_n(G')$ is easily obtained from Theorem 3.1.

Remark 2. The subgraph count $S_n(G)$ is a sum of dependent random variables, for which the exact calculation of the variance may be long and tedious. Our method approximates $S_n(G)$ by a sum of independent random variables, called the projection of $S_n(G)$, for which the variance calculation is elementary, providing the denominator of the normalized random variable in the conclusion of Theorem 3.1.

Remark 3. The upper end of the range of normal convergence does not depend on the graph G, as the lower end does, and is the best possible bound: If $n^2(1-p) \to c$ for some $0<c<\infty$, then $\binom{n}{2}(1-p) \to \frac{1}{2}c$, which implies that the number of edges of K_n which are absent in $K_{n,p}$ has a limiting Poisson distribution with mean $\frac{1}{2}c$. As a consequence, the number of subgraphs of K_n isomorphic to G which are not subgraphs of $K_{n,p}$, properly standardized, has an asymptotic Poisson distribution. If $n^2(1-p) \to 0$, then almost all graphs are complete, so the subgraph count is a deterministic function of n with probability tending to one.

Remark 4. For a star on d vertices, Theorem 3.1 provides normal convergence when $p = w_n n^{-1/(d-1)}$ where $w_n \to +\infty$. Since the star is strictly balanced with average degree $(d-1)/d$, the threshold function for existence of d-stars is $n^{-d/(d-1)}$, and normal convergence results of Karonski and Rucinski [15] apply when $p_n = w_n n^{-d/(d-1)}$ where $w_n \to \infty$ but $w_n = \sigma(n^{\delta})$ for any $\delta > 0$. The range between $n^{-1/(d-1)}$ and $n^{-d/(d-1)}$ is not covered by either result.

For a cycle of length d, Theorem 3.1 applies when $p_n = w_n n^{-1/(d-1)}$ where $w_n \to \infty$, and previous results apply for $p_n = w_n n^{-1}$ where $w_n \to \infty$ but $w_n = \sigma(n^{\delta})$ for any $\delta > 0$.

For the complete graph on d vertices, Theorem 3.1 gives

$$p_n = w_n n^{-1/[\binom{d}{2}-1]} = w_n n^{-2/[(d-2)(d+1)]},$$

where $w_n \to \infty$, as the lower bound on the range of normal convergence. However, the previous results give normal convergence for sequences near the threshold $n^{-2/(d-1)}$, again leaving a range of values where the asymptotic behavior is unknown. Rucinski [18] recently proved that normal convergence holds in this range.

4. *U*-statistic methodology

Consider a sequence $X_1, X_2, X_3, \ldots$ of independent identically distributed random variables. If h is a symmetric function of k variables, the *U-statistic* with kernel h based on n observations is defined by

$$U_n = \binom{n}{k}^{-1} \sum_{\{i_1,\ldots,i_k\} \in C(n,k)} h(X_{i_1}, X_{i_2}, \ldots, X_{i_k})$$

where $C(n, k)$ denotes the set of subsets of size k from $\{1, 2, \ldots, n\}$.

U-statistics were introduced by Hoeffding [11] as a generalization of the sample mean, and provide a class of unbiased estimators of distributional parameters in statistical estimation theory. Many common statistics, such as the sample mean, sample variance, and Wilcoxon test statistics, are U-statistics. The strong law of large numbers for U-statistics was established by Hoeffding [12] and Berk [1] using martingale methods. The central limit theorem was established in the original

paper of Hoeffding, and the rate of convergence to normality has been investigated by Grams and Serfling [9], Bickel [2], Chan and Wierman [7], Callaert and Jansen [6], and Helmers and Van Zwet [10]. U-statistics are useful as approximations to other classes of statistics, such as linear combinations of order statistics.

If each term $h(X_{i_1}, X_{i_2}, \ldots, X_{i_k})$ is weighted by a factor $w(i_1, i_2, \ldots, i_k)$, we have the more general form of a *weighted U-statistic*

$$W_n = \sum_C w(i_1, i_2, \ldots, i_k)h(X_{i_1}, X_{i_2}, \ldots, X_{i_k}).$$

If the weights $w(i_1, i_2, \ldots, i_k)$ take only 0 or 1 as values, the statistic W_n represents an 'incomplete' or 'reduced' U-statistic sum. Incomplete U-statistics are designed to be computationally simpler than the full sum, based on the reasoning that it should be possible to use fewer terms without much loss of information. Incomplete U-statistics have been investigated by Brown and Kildea [5], who proved asymptotic normality under certain balance conditions on the weights. Shapiro and Hubert [20] investigated asymptotic normality for weighted U-statistics.

The asymptotic behavior of a weighted U-statistic may often be determined using Hajek's projection method, approximating the weighted U-statistic by a sum of independent random variables. Define W_n^*, the projection of W_n, by

$$W_n^* = \sum_{i=1}^{n} E[W_n \mid X_i] - (n-1)E[W_n].$$

The original weighted U-statistic may be investigated by writing

$$W_n - E[W_n] = [W_n^* - E[W_n]] + [W_n - W_n^*],$$

and considering each term on the right side separately.

The projection W_n^* has the same mean value as W_n, so

$$W_n^* - E[W_n] = \sum_{i=1}^{n} [E[W_n \mid X_i] - E[W_n]]$$

is a sum of independent, identically distributed, mean zero random variables. Under appropriate conditions, its asymptotic distribution may be derived by a standard central limit theorem.

We will rely on a central limit theorem formulated for a double array of random variables $\{X_{nj}\}$, in which for each $n \geq 1$, there are k_n random variables $\{X_{nj}, 1 \leq j \leq k_n\}$, where it is assumed that $k_n \to \infty$ as $n \to \infty$. Denote the distribution function of X_{nj} by F_{nj}, and let

$$\mu_{nj} = E[X_{nj}]$$

$$\mu_n = E\left[\sum_{i=1}^{k_n} X_{nj}\right] = \sum_{j=1}^{k_n} \mu_{nj}$$

and
$$\sigma_n^2 = \text{Var}\left[\sum_{j=1}^{k_n} X_{nj}\right].$$

Theorem 4.1. *Let $\{X_{nj}: 1 \le j \le k_n\}$ be a double array in which the random variables in each row are independent. If the Lindeberg condition*

$$\frac{1}{\sigma_n^2}\sum_{j=1}^{k_n}\int_{|t-\mu_{nj}|>\varepsilon\sigma_n} (t-\mu_{nj})^2\, dF_{nj}(t) \to 0 \text{ as } n\to\infty$$

is satisfied for each $\varepsilon > 0$, then

$$\frac{\sum_{i=1}^{k_n}(X_{nj}-\mu_n)}{\sigma_n}$$

has an asymptotic standard normal distribution.

Note that the independence is assumed only within rows, but random variables in different rows may be arbtrarily strongly dependent.

To complete the analysis, one wishes to show that the error in the approximation, $W_n - W_n^*$, is negligible, so the asymptotic behavior of W_n is the same as that of W_n^*. This is often accomplished by the use of moment inequalities, such as Chebyshev's Inequality. If $\text{Var}(W_n - W_n^*) = o(\text{Var}(W_n^*))$, then by Chebyshev's inequality

$$P\left(\frac{|W_n - W_n^*|}{\text{Var}(W_n^*)} \ge \varepsilon\right) \le \frac{\text{Var}(W_n - W_n^*)}{\varepsilon^2\,\text{Var}(W_n^*)} \to 0$$

for every $\varepsilon > 0$, so $(W_n - W_n^*)/\sqrt{\text{Var}(W_n^*)}$ converges to 0 in probability, and the error in the approximation by the projection is negligible. Then the asymptotic normal distribution is obtained by writing

$$\frac{W_n - E[W_n]}{\sqrt{\text{Var}(W_n^*)}} = \frac{W_n^* - E[W_n]}{\sqrt{\text{Var}(W_n^*)}} + \frac{W_n - W_n^*}{\sqrt{\text{Var}(W_n^*)}} \xrightarrow{\mathscr{D}} N(0, 1),$$

by applying then central limit theorem to the projection term.

To aid in computing the variance of $W_n - W_n^*$, we now show that it may also be written in the form of a weighted U-statistic. Without loss of generality, we assume that $E[h(X_1, \ldots, X_k)] = 0$. The projection W_n^* is given by

$$\sum_{\{i_1,\ldots,i_k\}\in C(n,k)} w(i_1, \ldots, i_k)[g(X_{i_1}) + \cdots + g(X_{i_k})]$$

where

$$g(x) = E[h(X_1, \ldots, X_{k-1}, X_k)\,|X_1 = x].$$

Thus, we have the weighted U-statistic representation.

$$W_n - W_n^* = \sum_{C(n,k)} w(i_1, i_2, \ldots, i_k)\psi(X_{i_1}, X_{i_2}, \ldots, X_{i_k}),$$

where

$$\psi(X_1, X_2, \dots, X_k) = h(X_1, X_2, \dots, X_k) - \sum_{i=1}^{k} g(X_i).$$

The variance of $W_n - W_n^*$ is a weighted sum of expected values of products of the form

$$\psi(X_{i_1}, \dots, X_{i_k})\psi(X_{j_1}, \dots, X_{j_k}).$$

Note that if $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_k\}$ are disjoint, the expected value of the product is zero, by independence of the sets of random variables. In addition, if there is only one index in common, the expected value is still zero. To see this, compute

$$\begin{aligned} &E[\psi(X_{i_1}, \dots, X_{i_k})\psi(X_{i_1}, X_{j_2}, \dots, X_{j_k})] \\ &\quad = E[E[\psi(X_{i_1}, \dots, X_{i_k})\psi(X_{i_1}, X_{j_2}, \dots, X_{j_k}) \mid X_{i_1}, X_{j_2}, \dots, X_{j_k}]] \\ &\quad = E[\psi(X_{i_1}, X_{j_2}, \dots, X_{j_k})E[\psi(X_{i_1}, X_{i_2}, \dots, X_{i_k}) \mid X_{i_1}]] \end{aligned}$$

and use

$$\begin{aligned} &E[\psi(X_{i_1}, X_{i_2}, \dots, X_{i_k}) \mid X_{i_1}] \\ &\quad = E[h(X_{i_1}, \dots, X_{i_k}) - g(X_{i_1}) - \dots - g(X_{i_k}) \mid X_{i_1}] \\ &\quad = E[h(X_{i_1}, \dots, X_{i_k}) \mid X_{i_1}] - g(X_{i_1}) \\ &\quad = 0. \end{aligned}$$

Thus, only terms with two or more indices in common make a contribution to the variance of $W_n - W_n^*$.

By the Cauchy–Schwartz inequality, the contribution of any non-zero term is no greater than $E[\psi(X_1, \dots, X_k)^2]$, which is bounded by $(k+1)^2 E[h(X_1, \dots, X_k)]^2$.

5. Application to subgraph counts

Let G be a graph with k edges and l vertices, none of which are isolated, and consider the subgraph count

$$S_n \equiv S_n(G) = \sum_{\substack{A \subseteq K_n \\ |E(A)|=k}} I(A \sim G) \prod_{e \in E(A)} X(e).$$

By subtracting the mean, we obtain

$$S_n(G) - E[S_n(G)] = \sum_{\substack{A \subseteq K_n \\ |E(A)|=k}} I(A \sim G)\left\{\prod_{e \in E(A)} X(e) - p^k\right\}$$

which is a weighted U-statistic with kernel

$$h(X_1, \dots, X_k) = \prod_{i=1}^{k} X_i - p^k,$$

which has mean zero. The corresponding conditional expectation is

$$\begin{aligned}g(X_1) &= E\left[\prod_{i=1}^{k} X_i \mid X_1\right] - p^k \\ &= p^{k-1}X_1 - p^k,\end{aligned}$$

so the projection S_n^* has the form

$$S_n^* = \sum_{e\in K_n} a_n(p^{k-1}X_e - p^k)$$

where a_n denotes the number of subgraphs of K_n which contain a fixed edge and are isomorphic to G.

To compute a_n, note that there are $\binom{n-2}{l-2}$ different sets of l vertices containing the endpoints of the fixed edge. Thus, $a_n = \binom{n-2}{l-2}b_l$ where b_l denotes the number of subgraphs isomorphic to G on a set of l vertices [which without loss of generality we take to be $\{1, 2, \ldots, l\}$] with an edge between a fixed pair of vertices.

Let $C(n, G)$ denote the set of labelled subgraphs of K_n which are isomorphic to G. To find b_l, we count the number of edges in $C(l, G)$ by two different procedures. First, there are $\binom{l}{2}$ pairs of vertices, each having an edge in b_l subgraphs, for a total of $\binom{l}{2}b_l$ edges. Second, since there are $l!$ orderings of the vertices, there are $l!/a$ different subgraphs isomorphic to G, where a denotes the order of the automorphism group of G. Since each of these subgraphs has k edges, there is a total of $k(l!)/a$. Therefore,

$$b_l = \frac{k(l!)}{a\binom{l}{2}} = \frac{2k}{a}(l-2)!$$

so

$$S_n^* = \sum_{e\in K_n} \binom{n-2}{l-2}\frac{2k}{a}(l-2)!\,(p^{k-1}X_e - p^k).$$

Since the summands in S_n^* are independent and identically distributed,

$$\begin{aligned}\operatorname{Var}(S_n^*) &= \binom{n}{2}\operatorname{Var}\left(\binom{n-2}{l-2}\frac{2k}{a}(l-2)!\,p^{k-1}X_e\right) \\ &= \binom{n}{2}\binom{n-2}{l-2}^2\frac{4k^2}{a^2}[(l-2)!]^2p^{2k-2}p(1-p) \qquad (5.1)\\ &= \mathcal{O}(n^{2l-2}p^{2k-1}(1-p)] \quad \text{as } n\to\infty.\end{aligned}$$

Each of the summands in S^* is bounded by

$$B_n = \binom{n-2}{l-2}\frac{2k}{a}(l-2)!\max\{p^{k-1}-p^k, p^k\}.$$

If $B_n = o\sqrt{\operatorname{Var}(S_n^*)}$, then the Lindeberg condition in Theorem 4.1 is satisfied. If

p is a constant independent of n, then $B_n = \mathcal{O}(n^{l-2})$ and $\mathrm{Var}(S_n^*) = \mathcal{O}(n^{2l-2})$, so the Lindeberg condition is satisfied. If $p(n) \to 0$, $B_n = o\sqrt{\mathrm{Var}(S_n^*)}$ if and only if

$$\frac{n^{l-2}p^{k-1}\sqrt{1-p}}{n^{l-1}p^{k-1/2}\sqrt{1-p}} = \frac{1}{np^{\frac{1}{2}}} \to 0$$

i.e.

$$pn^2 \to \infty.$$

If $p(n) \to 1$, $B_n = o\sqrt{\mathrm{Var}(S_n^*)}$ if and only if

$$\frac{n^{l-2}p^k}{n^{l-1}p^{k-\frac{1}{2}}\sqrt{1-p}} \approx \frac{1}{n\sqrt{1-p}} \to 0$$

i.e. $(1-p)n^2 \to \infty$. When both of these convergence conditions hold, by the Lindeberg–Feller central limit theorem [Theorem 4.1], we have an asymptotic standard normal distribution for

$$\frac{S_n^* - E[S_n]}{\binom{n-2}{l-2}\frac{2k}{a}(l-2)!\sqrt{\binom{n}{2}p(1-p)}}.$$

To complete the proof of Theorem 3.1, we next show that $\mathrm{Var}(S_n - S_n^*) = o(\mathrm{Var}(S_n^*))$. This, by Chebyshev's inequality, implies that the error in the approximation of S_n by S_n^* is negligible. By the discussion in Section 4, $S_n - S_n^*$ may be represented as

$$\sum_{\{i,\ldots,i_k\}\subseteq\{1,\ldots,n\}} I(A \sim G)\varphi(X_{i_1}, \ldots, X_{i_k})$$

where

$$\psi(X_1, \ldots, X_k) = \left\{\prod_{i=1}^{k} X_i - p^k\right\} - \left\{\sum_{i=1}^{k} (p^{k-1}X_i - p^k)\right\}.$$

The terms in $S_n - S_n^*$ are uncorrelated unless there are two or more indices in common, which is equivalent to the corresponding subgraphs having two or more common edges.

For $d = 2, \ldots, k$, define

$$f_d = \left|\{(A, B) : |E(A) \cap E(B)| = d,\ A, B \in C(n, G)\right|.$$

i.e. f_d is the number of pairs of subgraphs isomorphic to G with exactly d common edges.

To compute f_d, decompose the set according to the number of common vertices, defining

$$f_d(i) = \left|\{(A, B) : |E(A) \cap E(B)| = d,\ |V(A) \cap V(B)| = i;\ A, B \in C(n, G)\}\right|$$

for $i = 3, 4, \ldots, l$ (since if A and B share two or more common edges, they have at least 3 common vertices). Then,

$$f_d = \sum_{i=3}^{l} f_d(i).$$

For each $i = 3, 4, \ldots, l$, we choose the i common vertices and $l - i$ additional vertices in each of A and B, so

$$f_d(i) = \binom{n}{i, l-1, l-i} e_d(i),$$

where $e_d(i)$ denotes the number of pairs (A, B) which can be obtained on two fixed sets of vertices $V_1 = V(A)$ and $V_2 = V(B)$ such that $|V_1 \cap V_2| = i$ and $|E(A) \cap E(B)| = d$. Since $e_d(i)$, $i = 3, 4, \ldots, k$, is a sequence of constants independent of n, we find that

$$\begin{aligned} f_d &= \sum_{i=3}^{k} \binom{n}{i, l-1, l-i} e_d(i) \\ &= \sum_{i=3}^{k} \binom{n}{2l-i} \binom{2l-1}{i} \binom{2l-2i}{l-1} e_d(i) \\ &= \mathcal{O}(n^{2l-3}). \end{aligned}$$

Therefore

$$\sum_{d=2}^{k} f_d = \mathcal{O}(n^{2l-3}).$$

Since

$$\begin{aligned} \operatorname{Var}(S_n - S_n^*) &\leq \Big(\sum_{d=2}^{k} f_d\Big) E[\psi^2(X_1, \ldots, X_k)] \\ &\leq \mathcal{O}(n^{2l-3}(k+1)^2 E[h^2(X_1, \ldots, X_k)]) \\ &= \mathcal{O}(n^{2l-3} p^k (1 - p^k)), \end{aligned}$$

we have

$$\begin{aligned} \frac{\operatorname{Var}(S_n - S_n^*)}{\operatorname{Var}(S_n^*)} &= \mathcal{O}\Big(\frac{n^{2l-3} p^k (1-p)^k}{n^{2l-2} p^{k-1} (1-p)}\Big) \\ &= \mathcal{O}\Big(\frac{1-p^k}{1-p} \frac{1}{np^{k-1}}\Big) \to 0 \end{aligned}$$

if $np^{k-1} \to \infty$.

Thus, under the hypotheses of Theorem 3.1, the projection is asymptotically normal and the error in the approximation is negligible, so the conclusion follows by the general theory discussed in Section 4.

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of their independent results presented here. Dr. Wierman thanks Michal Karoński for introducing him to the subgraph count problem.

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TOUGHNESS AND MATCHING EXTENSION IN GRAPHS

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1. Introduction and terminology

All graphs in this paper will be finite and connected and will have no loops or parallel lines.

Let n and p be positive integers with $n \leq \frac{1}{2}(p-2)$ and let G be a graph with p points having a perfect matching. Graph G is said to be *n-extendable* if every matching of size n in G extends to a perfect matching.

The concept of n-extendability for bipartite graphs was studied by Hetyei ([9]). But the study of the more general family of n-extendable graphs which are not necessarily bipartite seems to have even earlier roots. In the late 1950s, Kotzig [10–12] began to develop a decomposition theory for graphs with perfect matchings, but unfortunately these papers did not receive the attention that they deserve, due to the fact that they were written in Slovak. In the early 1960s, the study of decompositions of graphs in terms of their maximum matchings was begun by Gallai [7, 8] and independently by Edmonds [4]. One of the degenerate cases of their theory for *maximum* matchings, however, arises when the graphs in question have *perfect* matchings.

Motivated by these results of Kotzig, Hetyei, Gallai and Edmonds, Lovász [13] extended and refined the canonical decompositions already extant.

In this same paper, Lovász also introduced the concept of a *bicritical* graph. A graph G is said to be *bicritical* if $G-u-v$ has a perfect matching for every pair of distinct points u and v in $V(G)$. In the last ten years or so, the earlier work on decompositions of graphs in terms of their matchings has evolved further (see Lovász and Plummer ([15]) and today much attention continues to be focused upon the structure of bicritical graphs which are, in addition, 3-connected. Such graphs have been christened *bricks*. (See, for example, the paper by Edmonds et al. [5] and that of Lovász [14].)

But what is the connection between n-extendability and bicriticality? One of the results presented in Plummer [16] states that every 2-extendable graph is either bipartite or is a brick. (The reader should convince himself immediately

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that these two classes of graphs are disjoint.) Motivated by this result, the author has continued to study properties of n-extendable graphs (see [17–20]).

Let S be a point cutset in graph G and let $c(G-S)$ denote the number of components in $G-S$. Then, if G is not complete, the *toughness* of G is defined to be $\min |S|/c(G-S)$ where the minimum is taken over all point cutsets S of G, whereas we define the toughness of K_n to be $+\infty$ for all n. We denote the toughness of G by *tough*(G). We will also say that graph G is *k-tough* if $\text{tough}(G) \geq k$. This parameter was introduced by Chvátal [2, 3] who was initially motivated by studies about Hamiltonian cycles in graphs. He noted, however, in [2] that every 1-tough graph with an even number of points has a perfect matching.

A generalization of both the concepts of Hamiltonian cycle and perfect matching is the idea of a *k-factor* of a graph. A *k-factor* of a graph G is a spanning subgraph of G which is regular of degree k. Thus a perfect matching is just a 1-factor and a Hamiltonian cycle is just a connected 2-factor. Chvátal conjectured in [2] that if G is any graph on p points and if k is a positive integer such that G is k-tough and kp is even, then G has a k-factor. This conjecture has only recently been settled in the affirmative by Enomoto et al. [6].

In the present paper, we wish to treat some relationships between toughness of a graph and the n-extendability of the graph. In the next section we will prove two results. The first says essentially that if a graph has sufficiently high toughness (and has an even number of points) then it must be n-extendable. The second result applies to graphs with toughness less than one and presents an upper bound on the value of n for which such a graph can be n-extendable.

In the final section, we compare and contrast these results with the n-factor results of Enomoto et al.

Any graph terminology used, but not defined, in this paper may be found either in Bondy and Murty [1] or Lovász and Plummer [15].

2. Two results on toughness and n-extendability

In addition to the theorem of the author [16] mentioned in the Introduction, there are two other results proved in that paper which we shall use here and hence we begin by stating them without proof.

1980A. Theorem. *If $n \geq 2$ and G is n-extendable, then G is also $(n-1)$-extendable.*

1980B. Theorem. *If G is n-extendable, then G is $(n+1)$-connected.*

Our first result of the present paper follows in a straightforward way via Tutte's classical theorem characterizing graphs with perfect matchings.

2.1. Theorem. *Suppose that G is a graph with $p = |V(G)|$ points with p even. Let n be a positive integer with $p \geq 2n+2$. Then if tough$(G) > n$, graph G is n-extendable. Moreover, this lower bound on tough(G) is sharp for all n.*

Proof. First suppose that $n = 1$. Note that since tough$(G) \geq 1$, graph G has a perfect matching by Tutte's Theorem on perfect matchings.

Now suppose that for some line $e = xy \in E(G)$, line e lies in no perfect matching for G. Thus if $G' = G - x - y$, by the above-mentioned theorem of Tutte there is a set $S' \subseteq V(G')$ with $|S'| < c_o(G' - S')$. (Note that here $c_o(G' - S')$ denotes the number of components of $G' - S'$ which have an odd number of points.) But then by parity, $|S'| \leq c_o(G' - S') - 2$.

Now let $S_0 = S' \cup \{x, y\}$. Since G has a perfect matching, it follows that $c_o(G - S_0) \leq |S_0| = |S'| + 2 \leq c_o(G' - S')$. But $G - S_0 = G' - S'$ and so equality holds throughout and in particular, $c_o(G - S_0) = |S_0|$. (See Fig. 1.)

But now

$$\text{tough}(G) = \min_{S \subseteq V(G)} \frac{|S|}{c(G-S)} \leq \frac{|S_0|}{c(G-S_0)} \leq \frac{|S_0|}{c_o(G-S_0)} = 1,$$

contradicting the hypothesis of this theorem. So the desired conclusion holds when $n = 1$.

Now suppose $n \geq 2$, and assume that G is not n-extendable. Let $M = \{e_1, \ldots, e_n\}$ be a matching of size n which does not extend to a perfect matching. Denote $e_i = x_i y_i$ for $i = 1, \ldots, n$. Let $G_1 = G - x_1 - \cdots - x_n - y_1 - \cdots - y_n$. Hence G_1 has no perfect matching and thus by Tutte's Theorem, there is a set $S_1 \subseteq V(G_1)$ such that $|S_1| < c_o(G_1 - S_1)$. Hence by parity, $|S_1| \leq c_o(G_1 - S_1) - 2$. (Note that S_1 might be empty.)

Now $G_2 = G - x_n - y_n$ has a perfect matching since we have already proved that G is 1-extendable. Let $S_2 = S_1 \cup \{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$ and note that once

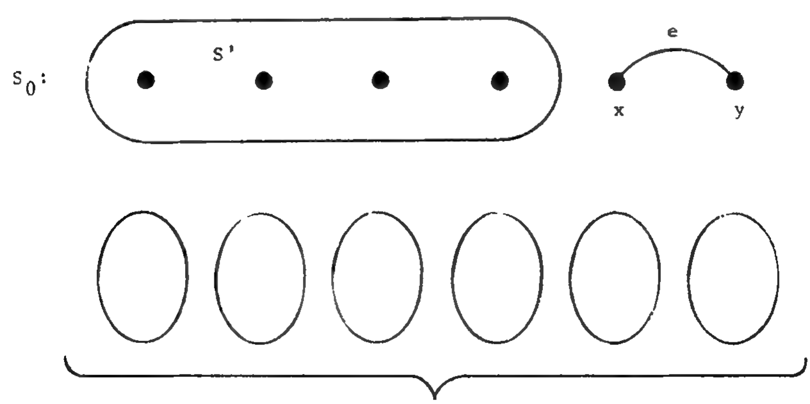

Fig. 1.

again by Tutte's Theorem, $c_o(G_2 - S_2) \leq |S_2| = |S_1| + 2n - 2 \leq c_o(G_1 - S_1) + 2n - 4$. But $G_2 - S_2 = G_1 - S_1$ and so it follows that $c_o(G_2 - S_2) \geq |S_1| + 2$.

Now let $S_3 = S_1 \cup \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Then, since $G - S_3 = G_2 - S_2$, we have

$$\begin{aligned}\text{tough}(G) &\leq \min_{S \subsetneq V(G)} \frac{|S|}{c(G-S)} \leq \frac{|S_3|}{c(G-S_3)} = \frac{|S_3|}{c(G_2 - S_2)}\\ &\leq \frac{|S_3|}{c_o(G_2 - S_2)} = \frac{|S_1| + 2n}{c_o(G_2 - S_2)}\\ &\leq \frac{|S_1| + 2n}{|S_1| + 2} \leq \frac{n\,|S_1| + 2n}{|S_1| + 2} = n,\end{aligned}$$

again a contradiction of the hypothesis.

It remains only to exhibit extremal graphs for each value of $n \geq 1$. For each positive integer $n \geq 1$, define graph H_n as follows. Join each of the $2n$ points of a copy of the complete graph on $2n$ points, $K_{2n}^{(1)}$, to each point of two disjoint copies of the complete graph K_{2n+1}. (See Fig. 2.) Then $|V(H_n)| = 6n + 2$ and it is easy to see that tough$(H_n) = n$. However, if N is any set of n independent lines in $K_{2n}^{(1)}$, then N does not extend to a perfect matching. $\square$

Now let us begin to think of some type of converse to the above result. Let us remark at the outset that if a graph G is n-extendable, *there is no lower bound on the toughness of* G. To see this, we construct the following family of graphs. Let n and k be any two positive integers. Let S be a set of $2n$ independent points and let graph $J(n, k)$ be constructed by joining each point of set S to both endpoints of each of $2n + k$ independent lines. (See Fig. 3.)

It is easy to verify that $J(n, k)$ is n-extendable for every value of k. Clearly, tough$(J(n, k)) \leq 2n/(2n + k)$, and hence tough$(J(n, k)) \to 0$ as $k \to \infty$. Of course the number of points in graph $J(n, k)$ is quite large and it makes sense to amend our search for some type of converse to Theorem 2.1 as follows. Again letting $p = |V(G)|$, we may ask if there is a function $f(p)$ such that if graph G is $f(p)$-extendable, then G is, say, 1-tough. The next result shows that the answer to this question is "yes".

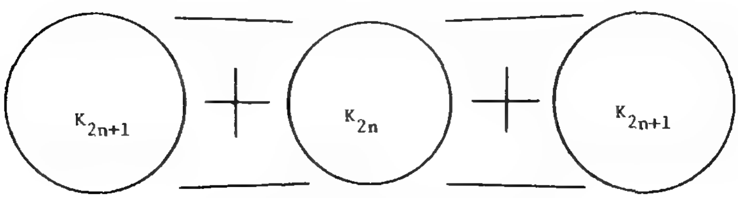

Fig. 2. The extremal family $\{H_n\}_{i=1}^{\infty}$.

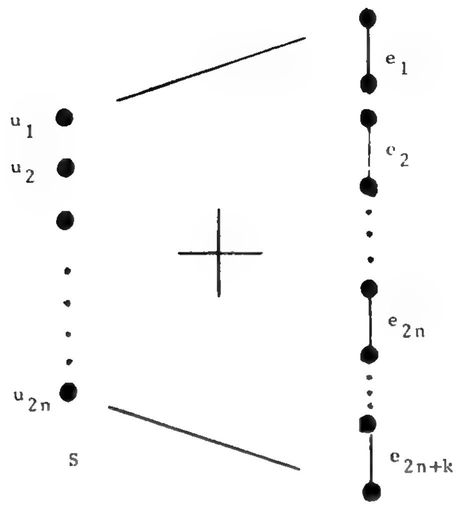

Fig. 3. The extremal family $\{J(n, k)\}_{n=1,k=1}^{\infty}$.

2.2. Theorem. *Let G be a graph with p points and let n be a positive integer. Suppose that G is n-extendable, but that tough$(G) < 1$. Then $n \leqslant \lfloor p - 2/6 \rfloor$ and this bound is sharp for each n.*

Proof. Since tough$(G) < 1$, there is a cutset S in G such that $G - S$ has more than $|S| = s$ components. Note that by Theorem 1980B, $s \geqslant n + 1 \geqslant 2$. Let the components of $G - S$ be $C_1, \ldots, C_{s+r}$, where $r \geqslant 1$.

Note that $G - S$ must have at least one even component, for if not, by Tutte's Theorem, G could not have a perfect matching, contradicting the hypothesis of the present theorem. So suppose that component C_1 is even and hence $|V(C_1)| \geqslant 2$. Since $n \geqslant 1$, G is 2-connected by Theorem 1980B, and hence there exists a line e_1 joining a point of C_1 to a point of S. By hypothesis, G is n-extendable and $n \geqslant 1$, so by Theorem 1980A, G is 1-extendable. So extend line e_1 to a perfect matching F_1 of G and note that by parity, F_1 matches at least two points of component C_1 into set S. It then follows that, in fact, $G - S$ must have at least 3 even components.

Claim 1. $G - S$ has at least n even components.

If $1 \leqslant n \leqslant 3$, we are done. So we may suppose that $n \geqslant 4$.

Suppose, to the contrary, that $G - S$ has t even components, where $3 \leqslant t \leqslant n - 1$. Relabeling these components if necessary, suppose that $C_1, \ldots, C_t$ are the even components of $G - S$. (Recall that altogether, $G - S$ has $s + r \geqslant s + 1 \geqslant n + 2$ components.)

We now construct a matching which contains two lines joining each of the

components $C_1, \ldots, C_t$ to different points of S. Let e_1 be any line joining C_1 to a point u_1 in S, relabeling the points of S if necessary. Now if all lines between S and C_2 are incident with point u_1, it then follows that u_1 is a cutpoint of G, a contradiction of the fact that G is 2-connected. Hence we can match points of C_1 and C_2 via lines e_1 and e_2 to distinct points u_1 and u_2 of S say, where once again we relabel the points of S if necessary.

Recall that $n \geq 4$. Suppose further that $C_1, \ldots, C_q$, $q < n$, have been matched into S to points $u_1, \ldots, u_q$ respectively. If we cannot match a point of $V(C_{q+1})$ into S at a point different from $u_1, \ldots, u_q$, then $\{u_1, \ldots, u_q\}$ is a cutset of G; that is, it separates C_{q+1} from all the other components of $G - S$. Hence $\kappa(G) \leq q < n$, contradicting the fact that (by Theorem 1980B) graph G is $(n+1)$-connected. Thus we have the matching of size n that we seek. Call it M_1.

Extend matching M_1 to a perfect matching F_2 of G. By parity, for each even component $C_1, \ldots, C_t$, matching F_2 must match at least one point to S other than that matched by M_1. Without loss of generality, let us suppose that a point of C_1 is matched to point $u_{n+1}, \ldots$, and that a point of C_t is matched to point u_{n+t}. (See Fig. 4.)

But now each of the odd components $C_{n+1}, \ldots, C_{s+r}$ must contain at least one

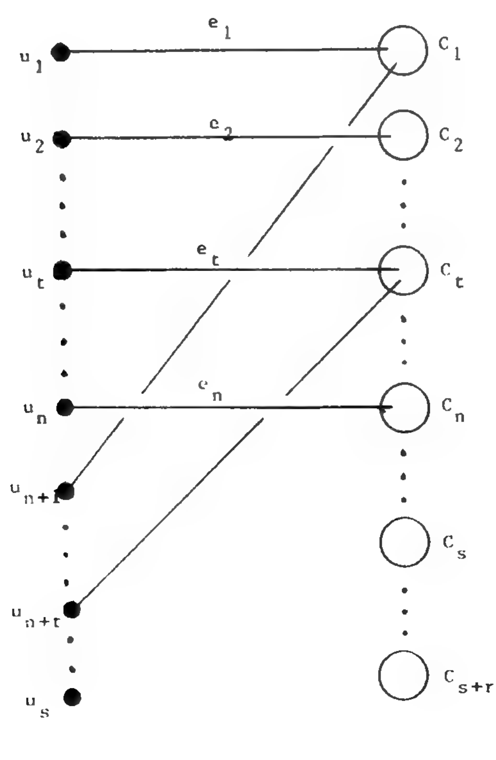

Fig. 4.

point which is matched by perfect matching F_2 into the set $\{u_{n+t+1}, \ldots, u_s\}$ of S. Thus it follows that $s+r-n \leqslant s-(n+t)$ and so $r \leqslant -t < 0$, a contradiction and Claim 1 is proved.

Finally, we prove

Claim 2. Graph $G-S$ has at least $2n+r$ even components.

By Claim 1, graph $G-S$ has at least n even components. Relabeling if necessary, suppose that they are $C_1, \ldots, C_n, \ldots, C_{n+\gamma}$. Just as in the proof of Claim 1, since G is n-connected, we can find a matching M_2 which joins exactly one point of component C_i to a point u_i in S for $i = 1, \ldots, n$, where yet again, we renumber points in S if necessary.

Since G is n-extendable, let us extend matching M_2 to a perfect matching F_3 of G. By parity, each of $C_1, \ldots, C_n$ has at least 2 points matched into S by F_3. So relabeling again if need be, assume that F_3 also matches a point u_{n+i} to a point in component C_i for $i = 1, \ldots, n$. (In particular at this point, we now know that $|S| = s \geqslant 2n$.)

Thus F_3 must match the remaining $s-2n$ points of S (if any) to some $s-2n$ points in $\bigcup_{i=1}^{s+r} V(C_i)$. So among the components $C_{n+1}, \ldots, C_{s+r}$, at least $(s+r-n)-(s-2n) = n+r$ must be even. These components, together with $C_1, \ldots, C_n$ give the $2n+r$ even components of $G-S$ as claimed.

Now we have

$$\begin{aligned} |V(G)| = p &= |S| + |V(C_1)| + \cdots + |V(C_{s+r})| \\ &\geqslant s + 2(2n+r) + (s+r-(2n+r)) \\ &= s + 4n + 2r + s - 2n = 2s + 2n + 2r \geqslant 6n + 2r \geqslant 6n + 2. \end{aligned}$$

So $n \leqslant (p-2)/6$ and since n is an integer, $n \leqslant \lfloor p-2/6 \rfloor$.

To show that the bound is sharp for all $n \geqslant 1$ consider the infinite family of graphs L_n where $L_n = J(n, 1)$ and $J(n, 1)$ is shown in Fig. 3. Note that $p = |V(L_n)| = 2n + 2(2n+1) = 6n+2$ and hence $n = (p-2)/6$ and it is easy to check that graph L_n is n-extendable, but not $(n+1)$-extendable. □

Of course, Theorem 2.2 can be restated as follows: if graph G is $(\lfloor p-2/6 \rfloor + 1)$-extendable, then G is 1-tough.

3. Comparisons with an n-factor theorem

Enomoto et al. [6] have proved the following result.

1985. Theorem. *Let G be a graph with at least $n+1$ points and suppose $tough(G) \geqslant n$. Then, if $n\,|V(G)|$ is even, G has an n-factor.*

This theorem answers in the affirmative a conjecture of Chvátal. In order to properly compare the conclusion of this result with that of our Theorem 2.1, let us try to state each result in as parallel a fashion as possible. Of course, if we were to define a graph to be "0-extendable" if it had a perfect matching, the two conclusions would say exactly the same thing when $n = 1$.

Now suppose $n \geq 2$ and consider the following two statements; the first being the result of Theorem 1985 and the second, our Theorem 2.1.

(A) $\text{tough}(G) \geq n \Rightarrow G$ has an n-factor.

(B) $\text{tough}(G) \geq n \Rightarrow G$ is $(n-1)$-extendable.

We claim that the two conclusions are independent, in that neither implies the other.

First consider the family of graphs $J(n, 1)$ already discussed above. Suppose $n \geq 2$. We already know that graph $J(n, 1)$ is n-extendable. Hence by Theorem 1980A it is also $(n-1)$-extendable. We claim it has no n-factor.

Suppose, to the contrary, that $J(n, 1)$ does have an n-factor, F. Then factor F must send $2n^2$ lines from set S to $G - S$. But each point of $G - S$ must send at least $n - 1$ lines to set S and hence we have at least $(n-1)(4n+2) = 4n^2 - 2n - 2$ lines of factor F from $G - S$ to S. But then $2n^2 \geq 4n^2 - 2n - 2$ and it follows that $n = 1$, a contradiction.

Finally, consider the infinite family of graphs $\{M_n\}_{n=4}^{\infty}$ constructed as follows. Graph M_n is formed by taking two copies of the complete graph K_{n+1} and joining corresponding points of the two copies with a perfect matching. (This is, of course, just the *prism* over K_{n+1}.) Graph M_n clearly has an n-factor consisting of precisely two components; namely, the two copies of K_{n+1}. On the other hand, we claim that graph M_n is not $(n-1)$-extendable. Let the points of the "top" K_{n+1} be $u_1, \ldots, u_{n+1}$ and the corresponding points of the "bottom" K_{n+1} be $v_1, \ldots, v_{n+1}$. (That is, u_iv_i, $i = 1, \ldots, n+1$, is the perfect matching joining the top and bottom.)

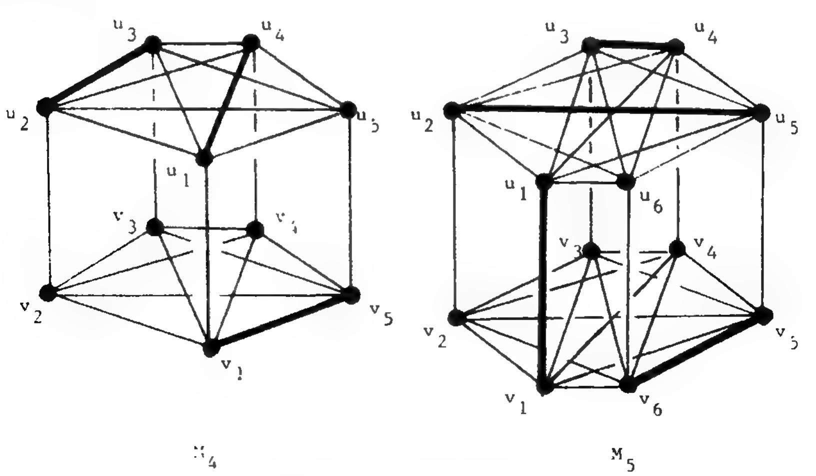

Fig. 5. Extremal graphs M_4 and M_5.

In order to prove our assertion, let us consider the cases for n odd and even separately.

First suppose n is odd and $n \geq 5$. Select the matching M_o consisting of u_1v_1, u_2u_n, $u_3u_{n-1}, \ldots$ together with line v_nv_{n+1}. Clearly M_o is a matching of size $n-1$, but it cannot be extended to a perfect matching for point u_{n+1} could never be matched in such a extension.

Now suppose $n = 3$. Let M_3 consist of two disjoint copies of K_4 joined by three independent lines. Clearly, graph M_3 is not 2-extendable, but it has a 3-factor.

Now suppose n is even. First let us suppose also that $n \geq 4$. In this case, select the matching M_e to consist of u_1u_n, $u_2u_{n-1}, \ldots$ together with line v_1v_{n+1}. Then matching M_e has size $n-1$, but it cannot be extended to a perfect matching since point u_{n+1} could never be matched.

Suppose $n = 2$. Let M_2 be the graph $K_4 - e$ for any line e in K_4. Clearly M_2 is not 1-extendable, but it has a 2-factor.

So in summary, the graphs $\{M_n\}$, for $n = 2, 3, \ldots$ have the properties that M_n has an n-factor, but is not $(n-1)$-extendable.

We show the graphs M_4 and M_5 in Fig. 5.

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BIPARTITE GRAPHS OBTAINED FROM ADJACENCY MATRICES OF ORIENTATIONS OF GRAPHS

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If G denotes a graph of order n, then the adjacency matrix of an orientation $\vec{G}$ of G can be thought of as the adjacency matrix of a bipartite graph $B(\vec{G})$ of order $2n$, where the rows and columns correspond to the bipartition of $B(\vec{G})$. For a graph H, let $k(H)$ denote the number of connected components of H. Set $m(G) = \min\{k(B(\vec{G})) : \vec{G}$ an orientation of $G\}$ and $M(G) = \max\{k(B(\vec{G}) : \vec{G}$ an orientation of $G\}$. R.A. Brualdi et al. [1] introduced these ideas and, among other results, proved that for a connected graph G of order n, $m(G) = M(G) = n + 1$ if and only if G is a tree. We prove an intermediate value theorem for $k(B(\vec{G}))$ and investigate the minimum and maximum number of edges possible in a graph G of order n for fixed $k(B(\vec{G}))$. In particular, we treat the case when G is complete, so that $\vec{G}$ is a tournament.

1.

If G is a graph, then $\vec{G}$ will denote a digraph obtained by orienting the edges of G and will be called an *orientation of* G. For example, if G is K_n, the complete graph with n vertices, then $\vec{G}$ is an n-tournament. O_G will denote the set of all orientations of G, $V(G)$ and $E(G)$ will denote the vertex set and edge set, respectively, of G, and $k(G)$ will denote the number of connected components of G. If $\vec{G} \in O_G$, $V(G) = V(\vec{G}) = \{u_1, \ldots, u_n\}$, and A is the adjacency matrix of $\vec{G}$ (i.e. $a_{ij} = 1$, if $u_i \to u_j$ in $\vec{G}$, and $a_{ij} = 0$, otherwise), then $B(\vec{G})$ denotes the bipartite graph with bipartition $V = \{v_1, \ldots, v_n\}$ and $W = \{w_1, \ldots, w_n\}$ where v_i is adjacent to w_j if $a_{ij} = 1$, $1 \leq i, j \leq n$ (i.e. $u_i \to u_j$ in $\vec{G}$). This paper concerns values of $k(B(\vec{G}))$ for various orientations $\vec{G}$ of a graph G and is motivated by questions raised by Brualdi et al. [1] in their treatment of these ideas.

Let $m(G) = \min\{k(B(\vec{G})) : \vec{G} \in O_G\}$ and $M(G) = \max\{k(B(\vec{G})) : \vec{G} \in O_G\}$. Brualdi et al. [1] proved that if G is a connected graph with n vertices, then $M(G) \leq n + 1$ with equality iff G is bipartite, and G is a tree iff $m(G) = M(G) = n + 1$. And, they raised the following problems:

(1) For a given graph G and integer t with $m(G) \leq t \leq M(G)$, does there exist an orientation $\vec{G} \in O_G$ so that $k(B(\vec{G})) = t$?
(2) What are the minimum and maximum number of edges in a graph with n vertices and given values of m and M?

We given an affirmative answer to the first problem and lay some ground-work for the second problem, namely for fixed n and k, $1 \leq k \leq 2n$, we establish the

maximum number of edges possible in a graph H with n vertices so that $k(B(\vec{H}))=k$ for some $\vec{H}\in O_H$ and the minimum number of edges possible in a graph G with n vertices so that $k(B(\vec{G}))=k$ for some $\vec{G}\in O_G$.

It is easy to see that a bipartite graph H with bipartition $\{v_1,\ldots,v_n\}$ and $\{w_1,\ldots,w_n\}$ can be expressed as $B(\vec{G})$ for some graph G if and only if H contains no edge v_iw_i $(1\leq i\leq n)$ and contains at most one of the edges in $\{v_iw_j, v_jw_i\}$ for all $1\leq i<j\leq n$.

First, some additional notation and terminology will be described. If $xy\in E(G)$, then $G-xy$ denotes the graph with vertex set $V(G)$ and edge set $E(G)-\{xy\}$. If $x\neq y$ belongs to $V(G)$ and x is not adjacent to y, then $G+xy$ will denote the graph with vertex set $V(G)$ and edge set $E(G)\cup\{xy\}$. If $x\in V(G)$, then $G-x$ denotes the graph with vertex set $V(G)-\{x\}$ and edge set $E(G)-\{xy: y$ is adjacent to $x\}$. Similar notations are used with $\vec{G}$ in place of G. It is well known that every tournament T can be *canonically decomposed* into (non-empty) strong components $A_1,\ldots,A_m$, for some $m\geq 1$, so that $V(A_1)\cup\cdots\cup V(A_m)$ is a partition of $V(T)$, and if $m>1$, then each vertex in A_i dominates each vertex in A_j iff $1\leq i<j\leq m$. If $m=1$, T is said to be *strongly connected* (or *strong*). If $m>1$, we call A_1 the *initial component* and A_m the *terminal component*. The corresponding vertex sets in $B(T)$ are denoted $B_1,\ldots,B_m$ and $C_1,\ldots,C_m$. A cycle using every vertex of a strong component is called a *spanning cycle* of that component. If $\vec{H}$ is used to denote an oriented graph, then H will always denote its underlying (undirected) graph. An arc from x to y will be denoted $x\to y$ or xy, and we will say that x *dominates* y or that y is *dominated* by x.

2.

An affirmative answer is given to problem 1 in the following result. The proof is due to Thomassen [2].

Theorem 1. *Let G be a graph and suppose that t is an integer satisfying $m(G)\leq t\leq M(G)$. Then there exists $\vec{G}\in O_G$ so that $k(B(\vec{G}))=t$.*

Proof. The result is clear if $|E(G)|=0$. So assume that $|E(G)|>0$. First, note that if $\vec{G}\in O_G$, if $u_1\to u_2$ is an arc in $\vec{G}$, and if $\vec{G}^*$ denotes $(\vec{G}-(u_1\to u_2))+(u_2\to u_1)$ (i.e. $\vec{G}^*$ is the result of reversing the orientation of $u_1\to u_2$), then $|k(B(\vec{G}^*))-k(B(\vec{G}))|\leq 1$. To see this, note that $B(\vec{G}^*)=(B(\vec{G})-v_1w_2)+v_2w_1$ (notation as in the introduction). Now, if v_1w_2 is a cut-edge of $B(\vec{G})$ and v_2 and w_1 are in the same component of $B(\vec{G})$, then $k(B(\vec{G}^*))=k(B(\vec{G}))+1$. If v_1w_2 is not a cut-edge of $B(\vec{G})$ and v_2 and w_1 are in different components of $B(\vec{G})$, then $k(B(\vec{G}^*))=k(B(\vec{G}))-1$. Otherwise, $k(B(\vec{G}^*))=k(B(G))$; so, $|k(B(\vec{G}^*))-k(B(\vec{G}))|\leq 1$.

Let $\vec{G}'$ (respectively, $\vec{G}''$) denote the orientation of G such that $m(G) = k(B(\vec{G}'))$ (respectively, $M(G) = k(B(\vec{G}''))$). $\vec{G}''$ can be obtained from $\vec{G}'$ by successively reversing the orientation of single arcs, and the number of components in the successive resulting bipartite graphs starts with $m(G)$ and ends with $M(G)$. By the first remark, these numbers either increase by 1, decrease by 1, or remain the same; that is, each number between $m(G)$ and $M(G)$ is assumed, so for some $\vec{G} \in O_G$, $k(B(\vec{G})) = t$. $\square$

3.

Now, we turn to the issues concerning problem 2. First, we study $k(B(T))$ for a tournament T. The results are stronger than necessary, but the next two lemmas are included for their own interest. As an immediate consequence we obtain the values of $m(K_n)$ and $M(K_n)$. The proof of Lemma 2 is straightforward and so is omitted.

Lemma 2. *Suppose that T is a nonstrong n-tournament, $n \geq 3$, with initial strong component A_1 and terminal strong component A_k, for some $k \geq 2$. Then*

$$k(B(T)) = \begin{cases} 1, & \text{if } |A_1| \geq 3 \text{ and } |A_k| \geq 3, \\ 2, & \text{if } (|A_1| = 1 \text{ and } |A_k| \geq 3) \text{ or } (|A_1| \geq 3 \text{ and } |A_k| = 1), \\ 3, & \text{if } |A_1| = |A_k| = 1. \end{cases}$$

Remark. It is easy to see that if T is the cyclic triple, then $k(B(T)) = 3$.

For strong n-tournaments, $n \geq 4$, we obtain the following result. Recall that if x is a vertex in a digraph D, then $d_D^+(x)$ (respectively, $d_D^-(x)$) denotes the number of vertices in D dominated by x (respectively, dominating x). The subscript D will be deleted if the context is clear.

Lemma 3. *Let T be a strong n-tournament, $n \geq 4$, then*

$$k(B(T)) = \begin{cases} 2, & \text{if there exists arc } yx \text{ in } T \text{ such that } d^+(x) = d^-(y) = n-2 \\ 1, & \text{otherwise.} \end{cases}$$

Proof. First, suppose T contains an arc yx in T such that $d^+(x) = d^-(y) = n - 2$. Suppose that a spanning cycle in T is given by $u_1u_2 \cdots u_nu_1$. Without loss of generality we may assume that $x = u_1$ and $y = u_n$. In $B(T)$, v_1 is adjacent to w_j, $2 \leq j \leq n-1$; each v_i, $2 \leq i \leq n-1$, is adjacent to w_n; and v_i is adjacent to w_{i+1}, $2 \leq i \leq n-1$. That is, $v_1, \ldots, v_{n-1}, w_2, \ldots, w_n$ are in one component of $B(T)$. And, edge v_nw_1 is the second component of $B(T)$. Note that the strong 4-tournament is such a T.

Now let T be an n-tournament, $n \geqslant 5$, containing no arc yx such that $d^+(x) = d^-(y) = n-2$. We employ induction to prove that $k(B(T)) = 1$. For $n = 5$, there are six strong n-tournaments, only four of which contain no arc yx as above (see [3, p. 92]); it is straightforward to check that $k(B(T)) = 1$ in these four instances (e.g. $B(T)$ is a 10-cycle if T is the regular 5-tournament). So, suppose that $n > 6$ and assume that the result holds for such strong tournaments of orders 4 through $n-1$. It is well known (e.g. see [3, p. 6]) that T contains a strong $(n-1)$-tournament, say denoted W. Let a spanning cycle in W be given by $u_2 u_3 \cdots u_n u_2$. Let u_1 denote the vertex of T not in W. If $k(B(W)) = 1$, then clearly $k(B(T)) = 1$ since $d^+(y_1) > 0$ and $d^-(u_1) > 0$. If $k(B(W)) \neq 1$, then by our induction hypothesis, W must contain an arc yx such that $d_W^+(x) = d_W^+(y) = n-3$. Without loss of generality we may assume that $x = u_2$ and $y = u_n$, and moreover, by the remarks above, we see that the edge $v_n w_2$ (of $B(W)$) is a single component of $B(W)$ and $v_2, \ldots, v_{n-1}, w_3, \ldots, w_n$ are the vertices of the second component of $B(W)$. Now, not both $u_2 u_1$ and $u_1 u_n$ are arcs of T, as otherwise arc $u_n u_2$ contradicts our assumption about T. The three remaining possibilities are treated in turn: (i) $u_2 u_1$ and $u_n u_1$ are arcs of T, (ii) $u_1 u_2$ and $u_1 u_n$ are arcs of T, (iii) $u_1 u_2$ and $u_n u_1$ are arcs of T. In case (i), since $v_2 w_1$ and $v_n w_1$ are edges in $B(T)$ which connect the two components of $B(W)$ and v_1 is adjacent to some w_j, $2 \leqslant j \leqslant n-1$ (since $d^+(u_1) > 0$), we see that $B(T)$ has one component. Case (ii) is similar to case (i) and utilizes the fact that $d^-(u_1) > 0$. In case (iii), if $u_3 u_1$ (respectively, $u_1 u_3$) is an arc of T, then $v_3 w_1 v_n w_2 v_1$ (respectively, $w_3 v_1 w_2 v_n w_1$) is a 4-path in $B(T)$ which connects the two components of $B(W)$ with vertices v_1, w_1. That is, in case (iii), $B(T)$ has one component. In any case, $k(B(T)) = 1$, and by induction the result follows. □

Theorem 4.

$$m(K_n) = \begin{cases} 1, & \text{if } n \geqslant 5, \\ 2, & \text{if } n = 1, 4, \\ 3, & \text{if } n = 2, 3, \end{cases}$$

and

$$M(K_n) = \begin{cases} 2, & \text{if } n = 1, \\ 3, & \text{if } n \geqslant 2. \end{cases}$$

4.

We now turn our attention to the following problem mentioned in Section 1 after problem (2). Fix $n > 1$. For each k, $m(K_n) \leqslant k \leqslant 2n$, what is the least possible number $f(n, k)$ of edges in a graph G with n vertices so that for some orientation $\vec{G}$ of G, $k(B(\vec{G})) = k$? We write $m(K_n)$ because $m(K_n) \leqslant k(B(\vec{G}))$ for all orientations of all graphs with n vertices; however, as we showed above $m(K_n) = 1$ for $n \geqslant 5$. It is easily seen that $f(n, k) = 2n - k$ for $1 \leqslant n \leqslant 4$.

Theorem 5. *For $n \geq 1$ and $m(K_n) \leq k \leq 2n$, $f(n, k) = 2n - k$.*

Proof. If G is a graph with n vertices and $k(B(\vec{G})) = k$ for some $\vec{G} \in O_G$, then

$$\begin{aligned}|E(G)| &= |E(B(\vec{G}))| = \sum \{|E(C)| : C \text{ a component of } B(\vec{G})\}\\ &\geq \sum \{(|V(C)| - 1 : C \text{ a component of } B(\vec{G})\}\\ &= 2n - k. \text{ So, } f(n, k) \geq 2n - k.\end{aligned}$$

To see that equality is possible a suitable graph is constructed. Suppose, first, that n is even, say $n = 2m$, for some $m \geq 3$. For $0 \leq j \leq 4m - 1$ let $\vec{G}_j$ be the oriented graph with vertex set $u_1, u_2, \ldots, u_{2m}$ and exactly the first j arcs in the sequence u_1u_2, u_3u_2, u_3u_4, $u_5u_4, \ldots, u_{2m-3}u_{2m-2}$, $u_{2m-1}u_{2m-2}$, $u_{2m-1}u_{2m}$, u_1u_3, $u_3u_5, \ldots, u_{2m-3}u_{2m-1}$, $u_{2m-1}u_1$, u_2u_4, $u_4u_6, \ldots, u_{2m-2}u_{2m}$, $u_{2m}u_2$ (the first $2m - 1$ terms describe an 'antidirected' spanning path, the next m terms describe a (directed) m-cycle, and the last m terms describe a (directed) m-cycle vertex disjoint from the previous one). Then it is easy to see that $B(\vec{G}_{4m-k})$ consists of a tree (actually a caterpillar) with $4m - k + 1$ vertices together with $k - 1$ isolates. Thus, if G_j denotes the underlying graph of $\vec{G}_j$, $1 \leq j \leq 4m - 1$, then G_{4m-k} has $n = 2m$ vertices, $2n - k = 4m - k$ edges, and $k(B(\vec{G}_{4m-k})) = k$. The case when n is odd is treated similarly. □

5.

A maximum version of the previous problem is the following problem: Fix $n \geq 1$. For each k, $m(K_n) \leq k \leq 2n$, what is the largest possible number $F(n, k)$ of edges in a graph G with n vertices so that for some orientation $\vec{G}$ of G, $k(B(\vec{G})) = k$? It is straightforward to check that $F(n, k) = f(n, k)$ for $1 \leq n \leq 3$, $m(K_n) \leq k \leq 2n$, and that

$$F(4, k) = \begin{cases} f(4, k), & \text{if } 5 \leq k \leq 8\\ f(4, k) + 1, & \text{if } 2 \leq k \leq 4.\end{cases}$$

Moreover, for $n \geq 5$, $F(n, 1) = F(n, 2) = F(n, 3) = \binom{n}{2}$, by Theorems 1 and 4. A lower bound for $F(n, k)$ is given next. For a real number r, we use $\{r\}$ (respectively, $[r]$) to denote the least integer greater than or equal to r (respectively, the greatest integer less than or equal to r).

Lemma 6. *Let $n \geq 5$ and $4 \leq k \leq 2n$. Then*

$$F(n, k) \geq \begin{cases} \binom{n}{2} - \binom{\left\{\frac{k-1}{2}\right\}}{2} - \binom{\left[\frac{k-1}{2}\right]}{2}, & \text{if } 4 \leq k \leq n,\\ \left(n - \left\{\frac{k-1}{2}\right\}\right)\left(n - \left[\frac{k-1}{2}\right]\right), & \text{if } n + 1 \leq k \leq 2n.\end{cases}$$

Proof. For n and k, $n \geq 5$ and $4 \leq k \leq 2n$, we define three vertex disjoint, oriented graphs A, B, C in order to construct an oriented graph $\vec{G}$ with n vertices such that $k(B(\vec{G})) = k$. If $4 \leq k \leq n$, let A be $\{(k-1)/2\}$ independent vertices, let B be a transitive $(n-k+1)$-tournament, and let C be $[(k-1)/2]$ independent vertices. For $4 \leq k \leq n$, $\vec{G}_1$ is constructed as follows: each vertex of A dominates each vertex in B and in C and each vertex of B dominates each vertex in C. Then $\vec{G}_1$ has

$$\binom{n}{2} - \binom{\left\{\frac{k-1}{2}\right\}}{2} - \binom{\left[\frac{k-1}{2}\right]}{2}$$

arcs, and $B(\vec{G}_1)$ consists of one big component (with $\{(k-1)/2\} + 2(n-k+1) + [(k-1)/2] = 2n-k+1$ vertices) and $k-1$ isolated vertices, i.e. $k(B(\vec{G}_1)) = k$. Now if $n+1 \leq k \leq 2n$, let A, B, and C be pairwise disjoint sets with $|A| = n - [(k-1)/2]$, $|B| = k-n-1$, and $|C| = n - \{(k-1)/2\}$, and form $\vec{G}_2$ with vertex set $A \cup B \cup C$ as follows: each vertex of A dominates each vertex in C. Then $\vec{G}_2$ has $(n - \{(k-1)/2\})(n - [(k-1)/2])$ arcs, and $B(\vec{G}_2)$ consists of one big component (with $|A| + |C| = 2n-k+1$ vertices) and $|A| + 2|B| + |C| = k-1$ isolated vertices, i.e. $k(B(\vec{G}_2)) = k$. $\square$

In the remainder of this paper we show that equality actually holds in Lemma 6. To do this we show (in Lemma 8) that if G is a graph with n vertices such that for some $\vec{G} \in \boldsymbol{O}_G$, $B(\vec{G})$ has k components, fewer than $k-1$ of which are isolates, then there is a graph H with $V(G) = V(H)$ and $|E(G)| \leq |E(H)|$ so that for some $\vec{H} \in \boldsymbol{O}_H$, $B(\vec{H})$ has k components and more isolates than does $B(\vec{G})$. This implies that there is a graph G with n vertices, $F(n,k)$ edges, and $k(B(\vec{G})) = k$ for some $\vec{G} \in \boldsymbol{O}_G$, such that $B(\vec{G})$ has $k-1$ isolates (and exactly one non-trivial component if $k \neq 2n$). Then equality in Lemma 6 will follow from the next result.

Lemma 7. *Let $n \geq 5$ and $4 \leq k \leq 2n$. Let $e(n,k)$ be the largest possible number of edges in a graph G with n vertices so that for some $\vec{G} \in \boldsymbol{O}_G$, $B(\vec{G})$ has k components, $k-1$ of which are isolates. Then*

$$e(n,k) = \begin{cases} \binom{n}{2} - \binom{\left\{\frac{k-1}{2}\right\}}{2} - \binom{\left[\frac{k-1}{2}\right]}{2}, & \text{if } 4 \leq k \leq n, \\ \left(n - \left\{\frac{k-1}{2}\right\}\right)\left(n - \left[\frac{k-1}{2}\right]\right), & \text{if } n+1 \leq k \leq 2n. \end{cases}$$

Proof. Clearly, if $k = 2n$, then $e(n,k) = 0$, so assume that $k < 2n$. Suppose that G is a graph with $V(G) = \{u_1, u_2, \ldots, u_n\}$. Recall that for an orientation $\vec{G}$ of G, our convention is to let those vertices of $B(\vec{G})$ in $V = \{v_1, v_2, \ldots, v_n\}$

(respectively, $W = \{w_1, w_2, \ldots, w_n\}$) correspond to the rows (respectively, columns) of the adjacency matrix of $\vec{G}$. Suppose that G is a graph so that $B(\vec{G})$ has k components, $k-1$ of which are isolates, for some orientation $\vec{G}$ of G and that $|E(B(\vec{G}))| = e(n, k)$. Since we assume that $k < 2n$, exactly one component C of $B(\vec{G})$ is not an isolate. Let X and Y denote the sets of isolated vertices contained in V and in W, respectively. And, for vertex w in $B(\vec{G})$, let $N(w)$ denote the sets of vertices adjacent to w in $B(\vec{G})$.

Suppose that $4 \leqslant k \leqslant n$. Since the number of isolates is $k-1 < n$, C contains vertices v_t and w_t for some t, $1 \leqslant t \leqslant n$. If for some i, $1 \leqslant i \leqslant n$, $v_i \in X$ and $w_i \in Y$, then $B(\vec{G}) - \{w_t v : v \in N(w_t)\} + \{w_i v : v \in N(w_t), v = v_i\}$ has k components, $k-1$ of which are isolates, and more edges than $B(\vec{G})$, a contradiction. Hence we may assume that for all i, $1 \leqslant i \leqslant n$, if $v_i \in X$, then $w_i \notin Y$; and that if $w_i \in Y$, then $v_i \notin X$. Thus, $|X| + |Y| = k-1$, and

$$e(n, k) = |E(B(\vec{G}))| \leqslant \binom{n}{2} - \binom{|X|}{2} - \binom{|Y|}{2} \leqslant \binom{n}{2}$$

$$- \binom{\left\{\frac{k-1}{2}\right\}}{2} - \binom{\left[\frac{k-1}{2}\right]}{2}$$

To see that equality is possible, see the construction of $\vec{G}_1$ in the proof of Lemma 6.

Suppose that $n+1 \leqslant k < 2n$. Since $n+1 \leqslant k$, there exists t, $1 \leqslant t \leqslant n$, with $v_t \in X$ and $w_t \in Y$. If, for some i, $1 \leqslant i \leqslant n$, both v_i and w_i are vertices in C, then

$$B(\vec{G}) - \{w_i v : v \in N(w_i)\} + \{w_t v : v \in N(w_i)\}$$

has at least $|E(B(\vec{G}))|$ edges and k components, $k-1$ of which are isolates. Hence, we may assume that if $v_i \in V(C)$, then $w_i \notin V(C)$, and that if $w_i \in V(C)$, then $v_i \notin V(C)$. Let $Z = \{i : v_i \in X, w_i \in Y\}$. Since $(|X| - |Z|) + |Z| + (|Y| - |Z|) = n$ and $|X| + |Y| = k-1$, $|Z| = k-1-n$. So, $(|X| - |Z|) + (|Y| - |Z|) = 2n - k + 1$. Hence,

$$e(n, k) = |E(B(\vec{G}))| \leqslant (|X| - |Z|)(|Y| - |Z|) \leqslant \left\{\frac{2n-2k+1}{2}\right\}\left[\frac{2n-2k+1}{2}\right].$$

To see that equality is possible, see the construction of $\vec{G}_2$ in the proof of Lemma 6.

The result follows. □

The reduction procedure referred to prior to Lemma 9 is established next.

Lemma 8. *Suppose that n and k are integers, $4 \leqslant k \leqslant 2n$. If G is a graph with n vertices such that for some $\vec{G} \in O_G$, $B(\vec{G})$ has k components, fewer than $k-1$ of which are isolated, then there exists a graph H with n vertices such that for some*

$\vec{H} \in O_H$, $B(\vec{H})$ also has k components, more isolates than $B(\vec{G})$, and at least as many edges as $B(\vec{G})$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$, let C be a nonisolate of $B(\vec{G})$, let δ_C be its characteristic function, and for $0 \leq \alpha$, $\beta \leq 1$ define $N_{\alpha\beta} = \{j : 1 \leq j \leq n,\ \delta_C(v_j) = \alpha,\ \delta_C(w_j) = \beta\}$. Of course, $|N_{10} \cup N_{11}| \geq 1$ and $|N_{01} \cup N_{11}| \geq 1$, as C is a non-isolate.

First suppose that $N_{11} \neq \emptyset$, say $s \in N_{11}$. If $\{v_i : i \in N_{01}\} \cup \{w_j : j \in N_{10}\}$ consists entirely of $|N_{01}| + |N_{10}|$ isolates, then all other non-isolate components of $B(\vec{G})$ have vertex sets entirely in $\{v_i : i \in N_{00}\} \cup \{w_j : j \in N_{00}\}$. In such a case suppose that v_i is a vertex of another non-isolate component $D \neq C$, where $i \in N_{00}$. Suppose v_i is adjacent to exactly $w_{j_1}, \ldots, w_{j_r}$ for some $r > 0$ and $j_1, \ldots, j_r \in N_{00} - \{i\}$. For each $j \in N_{10} \cup N_{11}$, none of $v_{j_1}, \ldots, v_{j_r}$ is adjacent to w_j. Delete the r edges $v_i w_{j_1}, \ldots, v_i w_{j_r}$ in $B(\vec{G})$ and add the $|N_{10} \cup N_{11}| \cdot r$ edges $\{v_j v_{j_l} : j \in N_{10} \cup N_{11},\ 1 \leq l \leq r\}$ to the resulting bipartite grap to obtain a suitable $B(\vec{H})$ as in the statement of the lemma (v_i is an isolate in $B(\vec{H})$, $V(C) \cup V(D) - \{v_i\}$ are the vertices of one component in $B(\vec{H})$, and otherwise components of $B(\vec{H})$ are as in $B(\vec{G})$). So, let us assume that $\{v_i : i \in N_{01}\} \cup \{w_j : j \in N_{10}\}$ does not consist entirely of isolates, say, without loss of generality, that w_j is in some non-isolate component $D \neq C$, where $j \in N_{10}$. Say w_j is adjacent in $B(\vec{G})$ to p vertices in $\{v_i : i \in N_{00}\}$, say $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$, and q vertices in $\{v_j : j \in N_{01}\}$, say $v_{j_1}, v_{j_2}, \ldots, v_{j_q}$. Recalling that $s \in N_{11}$, v_s may be adjacent in $B(\vec{G})$ to some of $w_{j_1}, \ldots, w_{j_q}$. If $q > 0$, choose notation so that for suitable $q_1 \geq 0$ and $q_2 \geq 0$ with $q = q_1 + q_2$, v_s is adjacent to each vertex in $\{w_j : j \in \{j_1, j_2, \ldots, j_{q_1}\}\}$ and v_s is not adjacent to each vertex in $\{w_j : j \in \{j_{q_1+1}, \ldots, j_{q_1+q_2} = j_q\}\}$. In $B(\vec{G})$ delete edges $v_{i_1} w_j, \ldots, v_{i_p} w_j$, add edges $v_{i_1} w_s, \ldots, v_{i_p} w_s$, for each $j \in \{j_{q_1+1}, \ldots, j_q\}$ add edge $v_j w_s$, delete edges $v_{j_1} w_j, \ldots, v_{j_q} w_j$ and add edges $v_j w_{j_1}, \ldots, v_j w_{j_q}$, and for each $j \in \{j_1, \ldots, j_{q_1}\}$ delete edge $v_s w_j$ and add edge $v_j w_s$, to obtain $B(\vec{H})$. Then w_j is an isolate of $B(\vec{H})$, $V(C) \cup V(D) - \{w_j\}$ of $B(\vec{H})$ is the vertex set of a component, and all other components are as in $B(\vec{G})$. Thus, $B(\vec{H})$ is a suitable bipartite graph. Note that this construction is valid if either $p = 0$ or $q = 0$ (however, $p + q > 0$). This completes the case in which $N_{11} \neq \emptyset$.

If any component C of $B(\vec{G})$ satisfies $N_{11} \neq \emptyset$, then proceed as above. So, suppose that for every component C of $B(\vec{G})$, $N_{11} = \emptyset$. Let C be a non-isolate component of $B(\vec{G})$. If $\{v_i : i \in N_{01}\} \cup \{w_j : j \in N_{10}\}$ consists entirely of $|N_{01}| + |N_{10}|$ isolates, then any other non-isolate component of $B(\vec{G})$ has vertex set entirely in $\{v_i : i \in N_{00}\} \cup \{w_j : j \in N_{00}\}$. In such a case, suppose that v_{i_0} is a vertex of another non-isolate component $D \neq C$, where $v_{i_0} \in N_{00}$. Suppose that in $B(\vec{G})$, v_{i_0} is adjacent to exactly $w_{j_1}, \ldots, w_{j_r}$, for some $r > 0$ and $j_1, \ldots, j_r \in N_{00} - \{i_0\}$. Note that $u_i, u_{j_1}, \ldots, u_{j_r}$ are independent in G and $\vec{G}$ since $i \in N_{10}$ and $j_1, \ldots, j_r \in N_{00}$ and w_i is an isolate in $B(\vec{G})$. In $B(\vec{G})$ delete edges $v_{i_0} w_{j_1}, \ldots, v_{i_0} w_{j_r}$, and for each $i \in N_{10}$, add the r edges $v_i w_{j_1}, \ldots, v_i w_{j_r}$, to obtain $B(\vec{H})$. Then v_{i_0} is an isolate in $B(\vec{H})$, $V(C) \cup V(D) - \{v_{i_0}\}$ is the vertex set of a non-isolate component in $B(\vec{H})$,

and otherwise components of $B(\vec{H})$ are as in $B(\vec{G})$. That is, $B(\vec{H})$ is a suitable bipartite graph. So, let us assume that $\{v_i : i \in N_{01}\} \cup \{w_j : j \in N_{10}\}$ does not consist entirely of isolates, say, without loss of generality, that w_j is in some non-isolate component $D \neq C$, where $j \in N_{10}$. Say w_j is adjacent in $B(\vec{G})$ to p vertices in $\{v_i : i \in N_{00}\}$, say $v_{i_1}, \ldots, v_{i_p}$, and q vertices in $\{v_j : j \in N_{01}\}$, say $v_{j_1}, \ldots, v_{j_q}$. Now, by the assumption made about components in $B(\vec{G})$ at the outset of this paragraph, none of $v_{j_1}, \ldots, v_{j_q}$ is adjacent to any of $w_{i_1}, \ldots, w_{i_p}$. Moreover, v_j is not adjacent to any of $w_{j_1}, \ldots, w_{j_q}$ (since w_j is adjacent to $v_{j_1}, \ldots, v_{j_q}$). But v_j is in C, so there exists j_0 in N_{01} so that v_j is adjacent to w_{j_0} (so w_{j_0} is in C) and $j_0 \neq j_1, \ldots, j_q$. And $u_{j_1}, \ldots, u_{j_q}, u_{j_0}$ are independent in G and $\vec{G}$, since $v_{j_1}, \ldots, v_{j_q}$ are in D and w_{j_0} is in C. If $q > 0$, then in $B(\vec{G})$ delete the p edges $v_{i_1}w_j, \ldots, v_{i_p}w_j$, add the p edges $w_{i_j}w_{j_1}, \ldots, v_{i_p}w_{j_1}$, delete the q edges $v_{j_1}w_j, \ldots, v_{j_q}w_j$ and add the q edges $v_jw_{j_1}, \ldots, v_jw_{j_q}$, and add the q edges $v_{j_1}w_{j_0}, \ldots, v_{j_q}w_{j_0}$, to obtain $B(\vec{H})$. Now, for $q > 0$, in $B(\vec{H})$ w_j is an isolate, $V(C) \cup V(D) - \{w_j\}$ is the vertex set of a component, and otherwise components are as in $B(\vec{G})$. Thus, $B(\vec{H})$ is as desired, provided $q > 0$. Now, suppose that $q = 0$, so that $p > 0$ (as w_j is in a non-isolate component D). Recall that $V(C) \cup V(D)$ contains none of $w_{i_1}, \ldots, w_{i_p}$ (by our assumption concerning $N_{11} = \emptyset$). In $B(\vec{G})$ delete the p edges $v_{i_1}w_j, \ldots, v_{i_p}w_j$, and for each $l \in \{i_1, \ldots, i_p\}$ proceed as follows: if either v_{j_0} is not adjacent to w_a or $v_{j_0}w_a$ is an edge whose removal doesn't disconnect v_{j_0} from w_a, then add edge $v_aw_{j_0}$ (and delete edge $v_{j_0}w_a$ if present); if $v_{j_0}w_a$ is an edge whose removal does disconnect v_{j_0} from w_a, then delete $v_{j_0}w_a$ and add edges $v_aw_{j_0}$ and v_jw_a. In the latter case that part of the component of $B(\vec{G})$ which contains w_a when $v_{j_0}w_a$ is deleted is incorporated with C into a new component by addition of edge v_jw_a. This construction yields a $B(\vec{H})$ for a suitable H as desired.

In any case, a suitable H is obtained from G. □

Combining Lemma 6, 7 and 8, we obtain our last result.

Theorem 9. *Let $n \geq 5$ and $4 \leq k \leq 2n$. Then*

$$F(n,k) = \begin{cases} \binom{n}{2} - \binom{\left\{\frac{k-1}{2}\right\}}{2} - \binom{\left[\frac{k-1}{2}\right]}{2}, & \text{if } 4 \leq k \leq n, \\ \left(n - \left\{\frac{k-1}{2}\right\}\right)\left(n - \left[\frac{k-1}{2}\right]\right), & \text{if } n+1 \leq k \leq 2n. \end{cases}$$

Proof. Let G be a graph with $F(n, k)$ edges and orientation $\vec{G}$ so that $B(\vec{G})$ has k components. If fewer than $k-1$ components of $B(\vec{G})$ are isolates, then by use of Lemma 8, repeatedly if necessary, we may suppose that we can find a graph H with some orientation $\vec{H}$ so that $B(\vec{H})$ has k components, $k-1$ of which are isolates, and $|E(G)| \leq |E(H)|$. By Lemma 8, $|E(H)| \leq e(n, k)$. Combining this

with Lemma 6 we have

$$e(n, k) \leqslant F(n, k) = |E(G)| \leqslant |E(H)| \leqslant e(n, k), \text{ i.e.}$$
$$F(n, k) = e(n, k). \quad \square$$

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C_n-FACTORS OF GROUP GRAPHS

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Let C_n denote the cycle of length n, $n \geqslant 3$. A C_n-factor of a graph G is a spanning subgraph of G in which every component is isomorphic to C_n. A graph G is C_n-factorable if it is the union of edge-disjoint C_n-factors. A spanning subgraph of G is a $\{C_a, C_b, \ldots, C_p\}$-factor of G if each of its components is isomorphic to one of the cycles $C_a, C_b, \ldots, C_p$. G is a $\{C_a, C_b, \ldots, C_p\}$-factorable if it is the union of edge-disjoint $\{C_a, C_b, \ldots, C_p\}$-factors. In this paper we present a sufficient condition for a group graph to have a C_n-factor and a sufficient condition for a group graph to be C_n-factorable. We also investigate $\{C_a, C_b, \ldots, C_p\}$-factorability in such graphs.

1. Introduction

A spanning subgraph F of a graph G is an $\{A, B, \ldots, P\}$-factor of G if each component of F is isomorphic to one of the graphs $A, B, \ldots, P$. In particular, if there exists a subgraph H of G such that every component of F is isomorphic to H, then F is said to be an *H-factor* of G. G is *$\{A, B, \ldots, P\}$-factorable* if it is the union of edge-disjoint $\{A, B, \ldots, K\}$-factors and the $\{A, B, \ldots, P\}$-factors are said to comprise an *$\{A, B, \ldots, P\}$-factorization* of G. G is *H-factorable* if it is the union of edge-disjoint H-factors and the H-factors are said to comprise an *H-factorization* of G. One of the np-problems mentioned by Akiyama and Kano in [1] is that of determining whether a given graph has a C_n-factor, where C_n denotes the cycle of length n, $n \geqslant 3$. We study this problem for the special case where G is a group graph. In this paper, we present a sufficient condition for a group graph to have a C_n-factor and a sufficient condition for a group graph to be C_n-factorable. We also investigate $\{C_a, C_b, \ldots, C_p\}$-factorability in such graphs.

A graph (G, R_A) with vertex set G and edge set R_A is a *group graph* if G is a group, A is a subset of G, and for $a, b \in G$, $(a, b) \in R_A$ if and only if $a^{-1}b \in A$. Since we wish to confine ourselves to finite, simple, connected and undirected group graphs in this paper, we require that G is finite, $e \notin A$ where e is the identity of G, $\langle A \rangle = G$ where $\langle A \rangle$ denotes the set of all finite products of elements of A, and $a \in A \Rightarrow a^{-1} \in A$, respectively. A discussion of the equivalence of these conditions to the given properties can be found in the textbook [2] by Teh and Shee.

2. C_n-Factors

Theorem 2.1. *If there exists $s \in A$ such that $o(s) = n \geqslant 3$, where $o(s)$ denotes the order of s in G, then (G, R_A) has a C_n-factor.*

Proof. Let $s \in A$ with $o(s) = n$ and let H_s denote the subgroup of G generated by s.

$$H_s = \{e, s, s^2, \ldots, s^{n-1}\},$$

where e is the identity of G. Let

$$G = a_0H_s \cup a_1H_s \cup a_2H_s \cup \cdots \cup a_kH_s$$

be the coset decomposition of G by H_s, where $a_0 = e$ and $k \geqslant 0$. The cosets in the decomposition are disjoint and each vertex of (G, R_A) belongs to exactly one coset.

To prove the theorem, it is sufficient to show that for any i, $0 \leqslant i \leqslant k$, there exists a cycle of length n which spans the vertices of (G, R_A) belonging to a_iH_s; and furthermore, that if $i \neq j$, then the cycle spanning the vertices in a_iH_s is disjoint from the cycle spanning the vertices in a_jH_s.

Consider the subgraph of (G, R_A) with vertex set

$$a_iH_s = \{a_i, a_is, a_is^2, \ldots, a_is^{n-1}\},$$

$0 \leqslant i \leqslant k$, and edge set

$$\{(a_i, a_is), (a_is, a_is^2), \ldots, (a_is^{n-1}, a_i)\}.$$

Clearly, this is a cycle of length n in (G, R_A). If $i \neq j$, $a_iH_s \cap a_jH_s = \emptyset$, hence the cycles

$$(a_i, a_is, a_is^2, \ldots, a_is^{n-1}, a_i)$$

and

$$(a_j, a_js, a_js^2, \ldots, a_js^{n-1}, a_j)$$

are disjoint. Thus the collection of cycles of length n obtained from a_iH_s, $0 \leqslant i \leqslant k$, comprises a C_n-factor of (G, R_A). □

This theorem implies that (G, R_A) has a C_n-factor for any n, $n \geqslant 3$, which is the order of some element of A. Note that an element of order two in A generates a P_2-factor of (G, R_A), where P_2 is the path with two vertices. In the literature, a P_2-factor is also known as a 1-factor, i.e. a factor in which all vertices have degree one.

Example 2.1. The dihedral group $D_n = \langle a, b \colon a^n = e,\ b^2 = e,\ (ab)^2 = e \rangle$ can always be used to produce a group graph with a C_n-factor, for any $n \geqslant 3$. In fact

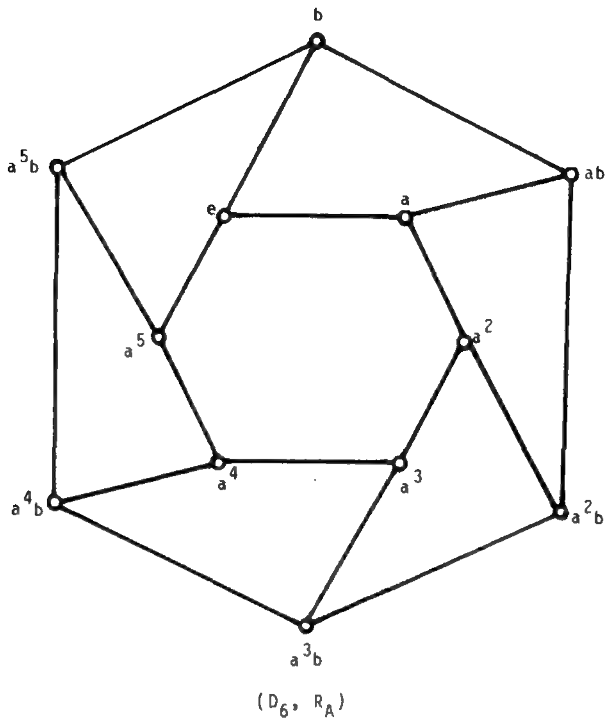

Fig. 1.

the group graph generated by D_n is $\{P_2, C_n\}$-factorable. With $n = 6$ and $A = \{a, b, a^5\}$, we obtain the group graph (D_6, R_A) shown in Fig. 1.

3. $\{C_a, C_b, \ldots, C_p\}$-Factorability

Theorem 3.1. *If $a, b, \ldots, p$ are the distinct orders of the elements of A and $a, b, \ldots, p \geq 3$, then (G, R_A) is $\{C_a, C_b, \ldots, C_p\}$-factorable.*

We first introduce some notation and prove lemmas which will simplify the proof of this theorem. If $s \in A$ with $o(s) = n \geq 3$, let $C_{[s]}$ denote the C_n-factor of (G, R_A) generated by s. Let s^{-1} denote the inverse of s in G.

Lemma 3.1. *$H_s = H_{s^{-1}}$ and $C_{[s]} = C_{[s^{-1}]}$.*

Proof. Since $s^{-1} = s^{n-1}$ and $-i \equiv n - i \pmod n$, we have $H_s = H_{s^{-1}}$ and $(a_i, a_is, a_is^2, \ldots, a_is^{n-1}, a_i) = (a_i, a_is^{-2}, \ldots, a_is^{-(n-1)}, a_i)$ for any $a_i \in G$. Hence the lemma follows. □

Lemma 3.2. *If $C_{[s]}$ and $C_{[t]}$ are not edge-disjoint, then $t=s$ or $t=s^{-1}$, in particular $o(s)=o(t)$.*

Proof. Let

$$C_{[s]}=\bigcup_{i=0}^{k}(a_i, a_is, a_is^2, \dots, a_is^{n-1}, a_i)$$

and

$$C_{[t]}=\bigcup_{j=0}^{l}(b_j, b_jt, b_jt^2, \dots, b_jt^{m-1}, b_j).$$

Suppose $C_{[s]}$ and $C_{[t]}$ are not edge-disjoint, then there exist i, j, q, r with $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant l$, $0 \leqslant q \leqslant n-1$, $0 \leqslant r \leqslant m-1$ such that

$$(a_is^q, a_is^{q+1})=(b_jt^r, b_jt^{r+1}).$$

This implies that either

(a) $a_is^q=b_jt^r$ and $a_is^{q+1}=b_jt^{r+1}$

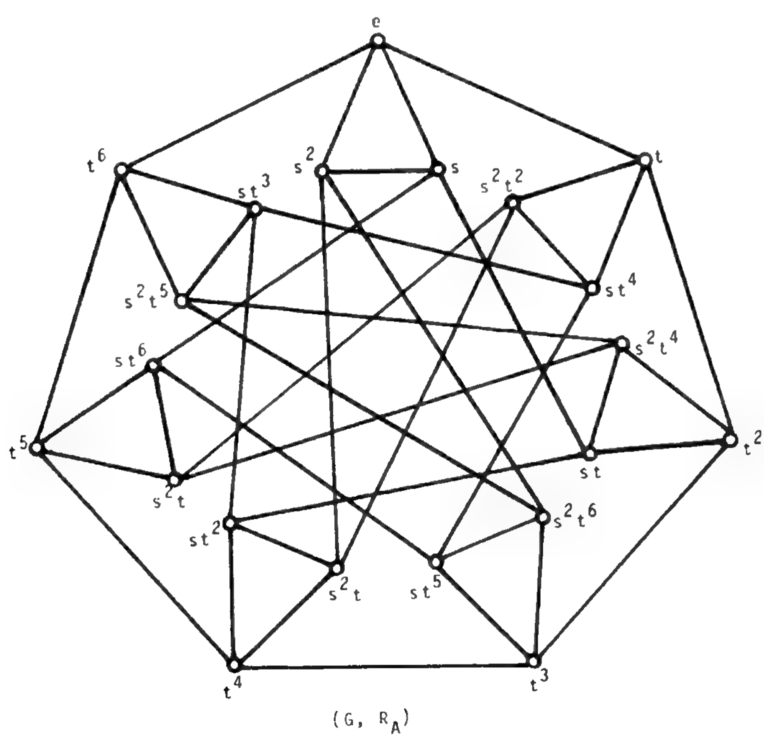

Fig. 2.

or

(b) $a_is^q = b_jt^{r+1}$ and $a_is^{q+1} = b_jt^r$.

From case (a), we have $a_is^{q+1} = (b_jt^r)t = (a_is^q)t$ or $s = t$. From case (b), we obtain $a_is^q = (b_jt^r)t = (a_is^{q+1})t$ or $e = st$. Hence $t = s^{-1}$. $\square$

We can now return to Theorem 3.1.

Proof of Theorem 3.1. By Theorem 2.1, each $s \in A$ generates a $C_{o(s)}$-factor of (G, R_A). By assumption, the components of this factor are isomorphic to one of the cycles $C_a, C_b, \ldots, C_p$. By Lemmas 3.1 and 3.2, either these factors are disjoint or they are equal. Hence, to prove that (G, R_A) is $\{C_a, C_b, \ldots, C_p\}$-factorable, it remains to show that every edge of (G, R_A) belongs to a $C_{o(t)}$-factor of (G, R_A) for some $t \in A$.

Let $(x, y) \in R_A$, then $x^{-1}y \in A$ which implies that $x^{-1}y = t$ for some $t \in A$. Hence $y = xt$ and $(x, y) = (x, xt)$. Since $C_{[t]}$ is a factor of (G, R_A) and $x \in G$, then

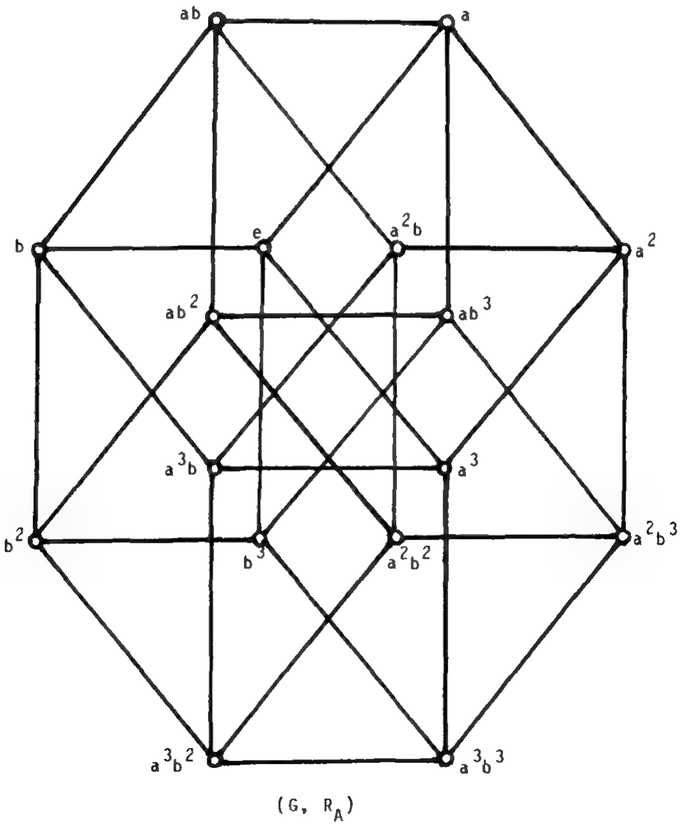

Fig. 3.

x is a vertex of $C_{[t]}$. Thus $x = b_j t^r$ for some j and r, $0 \leq j \leq l$, $0 \leq r \leq m-1$ and $xt = b_j t^{r+1}$. Finally, $(x, y) = (b_j t^r, b_j t^{r+1})$ which is an edge of $C_{[t]}$. $\square$

Example 3.1. In the group $G = \langle s, t : s^3 = e,\ sts^{-1} = t^2 \rangle$, $o(s) = 3$ and it can be shown that $o(t) = 7$. Using $A = \{s, t, s^2, t^6\}$, we obtain the $\{C_3, C_7\}$-factorable graph (G, R_A) shown in Fig. 2.

If some element of H has order two, then (G, R_A) is $\{P_2, C_a, C_b, \ldots, C_p\}$-factorable, where $a, b, \ldots, p$ are the distinct orders of the other elements of A.

We obtain a sufficient condition for (G, R_A) to be C_n-factorable as a corollary to Theorem 3.1.

Corollary 3.1. *If for every $s \in A$, $o(s) = n \geq 3$, then (G, R_A) is C_n-factorable.*

Since we consider only connected group graphs, this corollary actually requires that every generator of the group G be of order n. Note that if every generator of A is of order two, then (G, R_A) is P_2-factorable, or equivalently, is 1-factorable.

Example 3.2. The group $G = \langle a, b : a^4 = b^4 = e,\ ab = ba \rangle$ with $A = \{a, b, a^3, b^3\}$ generates the C_4-factorable group graph shown in Fig. 3.

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AN ALGORITHM FOR SOLVING THE JUMP NUMBER PROBLEM

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First, Cogis and Habib (*RAIRO Inform. Théor.* 13 (1979), 3–18) solved the jump number problem for series-parallel partially ordered sets (posets) by applying the greedy algorithm and then Rival (*Proc. Amer. Math. Soc.* 89 (1983), 387–394) extended their result to N-free posets. The author (*Order* 1 (1984), 7–19) provided an interpretation of the latter result in the terms of arc diagrams of posets explaining partly tractability of this special case.

In this paper, we present an algorithm for solving the jump number problem on arbitrary posets which extends the author's approach applied to N-free posets and makes use of two new types of greedy chains in posets introduced in companion papers [8, 9]. Complexity analysis of the algorithm supports our expectation that, going from N-free to arbitrary posets, the complexity of the problem increases with the number of dummy arcs required in their arc diagrams. The algorithm works in time which is linear in the poset size but factorial in the number of dummies, therefore it is a polynomial-time algorithm for posets with bounded number of dummies in their arc diagrams.

1. Preliminaries

A partially ordered set $(P, \leq)$ is in this paper simply written as P and called a *poset*. We assume also that all sets are finite. A *linear extension* of P is a total ordering $L = p_1 p_2 \cdots p_n$ of P preserving the relation, that is if $p_i < p_j$ in P then $i < j$. A pair (p_i, p_{i+1}) is a *jump* of L if $p_i \nless p_{i+1}$ in P. The jumps partition L into chains C_i of P, so we can write $L = C_0 + C_1 + \cdots + C_k$. The *jump number* $s(P)$ of P is equal to minimum k taken over all linear extensions L of P. A linear extension L of P is *optimal* if it has $s(P)$ jumps. The *jump number problem* consists in evaluating $s(P)$ and constructing an optimal linear extension. Although the problem is NP-complete (see Pulleyblank [4]), there exist polynomial-time algorithms for some special classes of posets defined by forbidden substructures (see Rival [5] and also [6]) or restricted by bounding some of their parameters (see [1] and [2]). The aim of this paper is to provide yet another algorithm of the latter type which generates special linear extensions introduced in [8].

We now define some basic graph-theoretic notions since we shall make a significant use of digraph representations of posets. A *digraph* may contain loops

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and multiple arcs, so it is defined as $D = (V, A, t, h)$, where V is the *vertex* set, A is the *arc* set, and t, h (t for *tail* and h for *head*) are two incidence mappings $t, h: A \to V$. An arc $a \in A$ is of the form $a = (t(a), h(a))$. A sequence of arcs $\pi = (a_1, a_2, \ldots, a_l)$ is a *path* of length l if $h(a_i) = t(a_{i+1})$ for $i = 1, 2, \ldots, l-1$. The path π *begins* with the arc a_1 and also with the vertex $t(a_1)$. Similarly, π *terminates* with a_l and with $h(a_l)$. Therefore, we can write $t(\pi) = t(a_1)$ and $h(\pi) = h(a_l)$. A digraph is *acyclic* if it contains no path of length greater than 1 and such that $h(a_l) = t(a_1)$. The *transitive closure* of D is denoted by $\operatorname{tc} D = (V, \operatorname{tc} A, t^*, h^*)$, where $(a_1, a_2, \ldots, a_l)$, $l \geq 1$, is a path in D if and only if $\operatorname{tc} A$ contains an arc b such that $t^*(a_1) = t^*(b)$ and $h^*(a_l) = h^*(b)$. Let us denote $A^* = \operatorname{tc}(A) \cup \{(v, v): v \in C\}$.

Posets can be represented by graphs and digraphs. In this paper we make a significant use of digraph representations (called arc diagrams) in which the poset elements are assigned to arcs and the relation is preserved along the paths of the digraphs. Formally, an *arc diagram* of a poset P is an acyclic digraph $D^A(P) = (V, R, t, h)$ without loops (but possibly with parallel arcs) and a mapping $\phi: P \to R$ such that for every $p, q \in P$, $p \neq q$ we have

$$p < q \text{ in } P \text{ iff } (h^*(\phi(p)), t^*(\phi(q))) \in R^*,$$

where t^*, h^* are the incidence mappings of $\operatorname{tc} D^A(P)$ and $R^* = \operatorname{tc}(R) \cup \{(v, v): v \in V\}$. For the sake of simplicity, we shall denote an arc diagram by a digraph $D^A(P)$ in which certain arcs (drawn in solid lines) are labelled by the poset elements in the way which preserves the relation between them along the paths in $D^A(P)$. Let us denote $S = R - \phi(P)$. An arc $a \in \phi(P)$ is a *poset arc* and otherwise a is called a *dummy arc* (and drawn in dotted lines). A path π in $D^A(P)$ is a *poset path* if it consists entirely of poset arcs. For the sake of completeness we define also the *vertex diagram* (known also as a *Hasse diagram*) of P to be a digraph $D^H(P) = (P, A)$ in which $(p, q) \in A$ if and only if q covers p in P. (Note that, since vertex diagrams contain no parallel arcs, we can drop out incidence mappings from their definition.) Figure 1 shows vertex and arc diagrams of some sample posets.

Arc diagrams of posets are best know in network analysis as event or PERT networks, where they are used mainly for time analysis of projects whose precedence relations are represented by activity-one-arc networks. In poset theory however, arc diagrams have not been widely accepted as poset representations. The main reason behind this is that, in contrast to the uniqueness of vertex diagrams, a poset may have an infinite number of arc diagrams and in fact there is no standard one. Moreover, finding an arc diagram with the minimum number of dummies (which may be blamed for non-popularity of arc diagrams) is an NP-complete problem in general.

Recently, it was demonstrated by Möhring [3] that certain hard combinatorial problems can be efficiently solved on comparability graphs by using arc diagrams of corresponding posets. For the jump number problem, arc diagrams have been

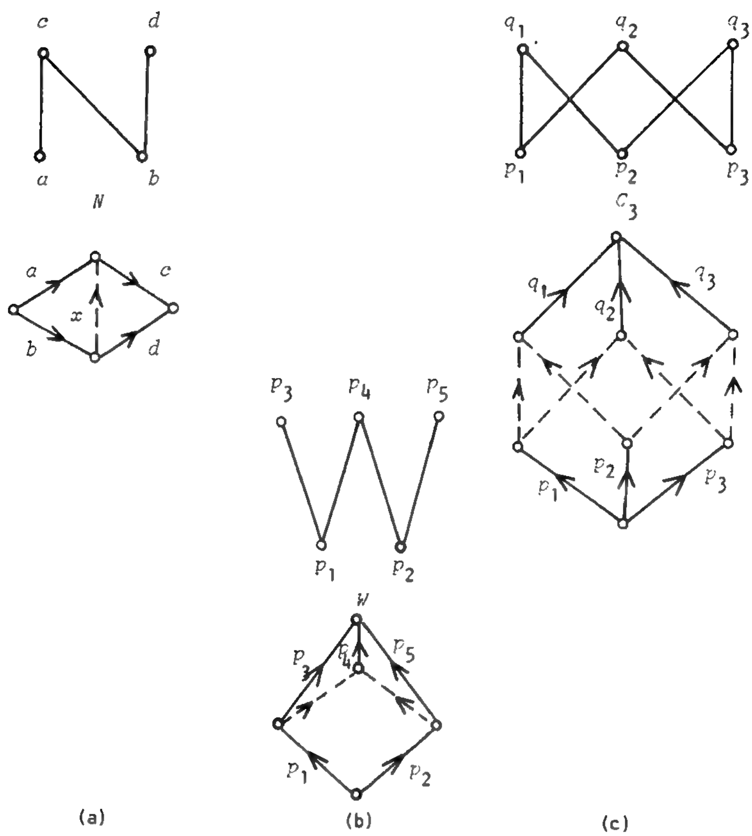

Fig. 1. Vertex and arc diagrams of some posets.

used in [6] to simplify the solution and derive the number of jumps in N-free posets. (N-free posets have arc diagrams with no dummy arcs, see Section 4 in [7].) Then, this approach has been also used in [7] to attack the problem for arbitrary posets. Finally in [8], a new proper subfamily of linear extensions has been defined on an arc diagram of a poset that contains an optimal solution and significantly reduces the search space. The result of [8] have been reformulated and improved in [9], where considerations are carried on directly on posets.

The purpose of this paper is to present an implementation of the results of [8] and [9], and discuss its complexity.

2. Greedy paths and linear extensions

We assume in the sequel that a poset P is represented by an arc diagram $D^A(P)$ which:

(i) has exactly one source and exactly one sink, and

(ii) for every dummy arc a, $\text{indeg}(h(a)) > 1$ and $\text{outdeg}(t(a)) > 1$.

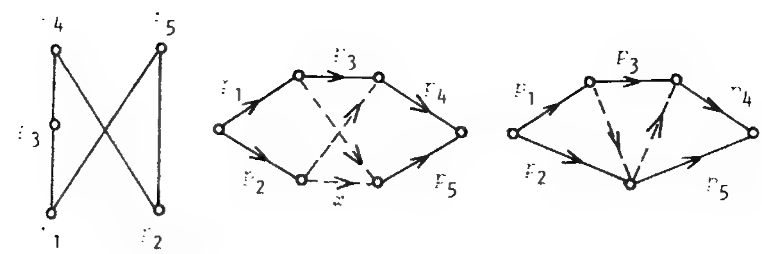

Fig. 2. Poset and its two compact arc representations.

An arc diagram $D^A(P)$ which satisfies conditions (i) and (ii) is called *compact.* Note that we weakened the conditions imposed on arc diagrams to be compact formulated in [8]. For instance, both arc diagrams shown in Fig. 2 are compact, whereas the first one is not compact in the sense defined in [8] since dummy arc x can be contracted.

An arbitrary arc diagram of a poset can be easily transformed to its compact form. Moreover, an arc diagram which has minimum number of vertices is compact and can be produced for every poset by a polynomial-time algorithm, we refer the reader to Section 4 of [7] for details.

A chain C of a poset P is *greedy* if there are no elements $p \in P - C$ and $q \in C$ such that $p < q$ and moreover for no r which covers $\sup C$ in P, the chain $C \cup \{r\}$ has this property. A linear extension $L = C_0 + C_1 + \cdots + C_s$ of P is *greedy* if C_i is a greedy chain in $P - \bigcup_{j<i} C_j$. It is easy to see that every poset has an optimal linear extension which is greedy. It was proved in [8] that for every poset the class of greedy linear extensions can be further restricted to the class of *semi-strongly greedy* ones which contains an optimal one. The purpose of this paper is to provide an efficient algorithm for generating an optimal semi-strongly greedy linear extension of a poset.

A semi-strongly greedy linear extension consists of two types of greedy chains which are naturally defined in the terms of arc diagrams. Let $D^A(P) = (V, R, t, h)$ be a compact arc diagram of a poset P. A path $\pi = (x_1, x_2, \ldots, x_k)$, $k \geq 1$ of $D^A(P)$ is called a *greedy path* if it has the following properties:

1. no vertex of π except $h(x_k)$ is a head of any arc $y \in R$, $y \neq x_j$ for every j, $0 \leq j \leq k$;
2. x_k is a poset arc; and
3. π cannot be extended to a path which satisfies 1 and 2, and has more poset arcs than π has.

Let C_π denote the sequence of poset arcs of the path π. It is easy to see that a greedy path π in a compact arc diagram $D^A(P)$ of P contains no dummy arcs. Moreover, the corresponding C_π is a greedy chain in P. On the other hand, every greedy chain C of P generates a greedy path in a compact arc diagram $D^A(P)$ – let $\pi(C)$ denote the corresponding path. Thus, we have a one-to-one correspondence between greedy chains of a poset and greedy paths in its compact

arc diagram. Therefore, we can use greedy paths and greedy chains interchangably.

A poset is *N-free* if its vertex diagram contains no induced subgraph isomorphic to the vertex digram N – see Fig. 1(a). N-free posets are precisely those which admit arc diagrams with no dummy arcs (see [6]). In what follows we assume that an N-free poset is represented by such a diagram.

If a poset P is N-free then every greedy linear extensions L of P is optimal (Rival [5]). In the other words, an optimal linear extension of an N-free poset P may begin with an arbitrary greedy path of $D^A(P)$. We now identify some greedy chains in arbitrary posets which share this property.

A greedy path π in a compact arc diagram $D^A(P)$ of a poset P is called *strongly greedy* if, additionally to conditions 1–3, π satisfies:

4. either (a) $h(\pi)$ is the sink of $D^A(P)$,
   or (b) $h(\pi)$ is the head of a poset arc b ($b \neq a_k$) such that every path terminating with b has no vertex which is incident with a dummy arc.

In particular, π is strongly greedy if π is greedy and $h(\pi)$ terminates a greedy path π' ($\pi' \neq \pi$) whose no vertex is a tail of a dummy arc. The diagram in Fig. 1(c) shows that not every poset has a strongly greedy path. The main property of strongly greedy paths is described in the following theorem.

Theorem 1 [8]. *If $D^A(P)$ admits a strongly greedy path π then there exists an optimal linear extension of P which begins with C_π.*

We proved in fact that for a strongly greedy path π in $D^A(P)$, every greedy linear extension L of P can be transformed to another linear extension L^* of P which begins with C_π and L^* has no more jumps than L. The examples in [8] show that such a transformation may result in decreasing the number of jumps in linear extensions. Since every N-free poset P admits a strongly greedy path π and a subposet $P - C_\pi$ of P is also N-free, Theorem 1 provides another proof of the formula for the jump number of N-free posets, see [6] and [8].

If $D^A(P)$ contains no strongly greedy paths then $D^A(P)$ contains greedy paths which in a certain sense are better than the other. Note that if π is a greedy path and π contains a vertex v which is a tail of at least one dummy arc but not a head of any dummy arc then taking C_π to a linear extension of P results in removing from $D^A(P)$ (together with π) all dummy arcs incident with v. More precisely, a greedy path π is called *semi-strongly greedy* if:

4′. π contains a vertex which is a tail of a dummy arc but not a head of any dummy arc.

For instance the path (b, d) in Fig. 1(a) and the paths (p_1), (p_2) and (p_3) in Fig. 1(c) are semi-strongly greedy. Note that all strongly greedy paths in diagrams of Figs. 1(a) and (b) are also semi-strongly greedy. An arc diagram may not contain a semi-strongly greedy path, e.g. when it has no dummy arcs. It is

however easy to show that every arc diagram which has no strongly greedy paths contains a semi-strongly greedy path. A formal proof of this fact given in [8] is based on elementary properties of acyclic digraphs formed by dummy arcs in compact arc diagrams.

A counterpart of Theorem 1 for semi-strongly greedy paths has a weaker form and says that in the absence of strongly greedy paths in $D^A(P)$, a good candidate to begin an optimal linear extension is one of the semi-strongly greedy paths in $D^A(P)$. Formally we have

Theorem 2 [8]. *If an arc diagram $D^A(P)$ of P contains no strongly greedy paths then P has an optimal linear extension which begins with a semi-strongly greedy path.*

Theorems 1 and 2 guarantee that every poset has an optimal linear extension $L = C_0 + C_1 + \cdots + C_s$, herafter called *semi-strongly greedy* such that each chain C_i is strongly greedy in $P_i = P - \bigcup_{j<i} C_i$ or semi-strongly greedy in P_i if P_i has no strongly greedy chains. In the next section we present an algorithm which finds an optimal linear extension of a poset among its semi-strongly greedy ones.

We conclude this section with a lower bound to the jump number which is also suggested by arc diagrams.

Observation 1. *If $D^A(P) = (V, R, t, h)$ is an arc diagram of a poset P then*

$$\sum_{v \in V} \max\{0, \mathrm{index}_P(v) - 1\} \leq s(P), \tag{1}$$

where $\mathrm{indeg}_P(v)$ *is the number of poset arcs coming into* v.

Proof. Note first that if $\mathrm{indeg}_P(v) > 1$ $(v \in V)$ then $\mathrm{indeg}_P(v)$ cannot be decreased by any greedy path which does not terminate at v. On the other hand, the removal of a greedy path from $D^A(P)$ that does not contain v results in a compact arc diagram in which $\mathrm{indeg}_P(v)$ is unchanged. Hence, only a greedy path terminating at v may reduce $\mathrm{indeg}_P(v)$ and each such a path contributes one to the jump number of P. This proves the bound. □

The bound (1) can be further improved by adding 1 to the left hand side if, after removing strongly greedy paths from $D^A(P)$, the reduced diagram admits only semi-strongly greedy paths π whose terminal vertices $h(\pi)$ are of total indegree 1 (in this case, the only other arcs incident with $h(\pi)$ are dummies which go out of $h(\pi)$).

3. An algorithm

A search method for finding for a poset a semi-strongly greedy linear extension with the minimum number of jumps is implemented in procedure OPTLINEXT.

A poset P is given by its compact arc diagram $D = (V, R, t, h)$. Such a diagram can be produced by a polynomial-time algorithm which constructs arc diagrams with minimum number of vertices, see Section 4 of [7] for a survey of construction methods of arc diagrams. An arbitrary arc diagram can be easily transformed to a compact form by contracting dummy arc a for which $\text{indeg}(h(a)) = 1$. Such a transformation can be applied in the algorithm whenever a current diagram (which is compact) is reduced by the removal of a (strongly or semi-strongly) greedy path. The diagram is represented by adjacency lists of arcs coming to and going out of each vertex. It is convenient to partition each list into two sublists of poset and dummy arcs, respectively.

Greedy paths (strongly and semi-strongly) in D can be easily identified by inspecting the vertices of D in a topological order, that is, when visiting vertex v we can be sure that all its predecessors have been already visited. To keep track of greedy paths, we define two queues S and W which store strongly and semi-stongly greedy paths in the current diagram. The paths in S and W are uniquely represented by their end-arcs. Statement 1 in the procedure is assumed to find all these quantities in the given arc diagram and then procedure REMOVE takes care of updating S and W. The number of jumps in a linear extension L of P is denoted by $S(L, P)$.

procedure OPTLINEXT(arc diagram: D, linear extension: **var** L^*);
  {The procedure finds an optimal semi-strongly greedy linear extension L^* of a poset p represented by its arc diagram D.}
procedure REMOVE(α, **var** D, S, W);
  {This procedure removes greedy path α from arc diagram D, transforms $D - \alpha$ into a compact form and updates queues S and W of strongly and semi-strongly greedy paths in the current arc diagram. Implementation details of this procedure are left to the reader.}
procedure SUBLINEXT(π, L, D, S, W);
  {The procedure extends L with a semi-strongly greedy path π removes π from D, extends L with strongly greedy paths which may result and then attempts to repeat this process.}
begin
  augment L with π;
  REMOVE(π, D, S, W);
  while $S \neq \emptyset$ **do begin**
    $\delta \leftarrow S$; augment L with δ; REMOVE(δ, D, S, W) **end**;
  if $D \neq \emptyset$ **then**
    for every ρ in W **do**
      SUBLINEXT(ρ, L, D, S, W)
  else if $s(L, P) < r$ **then begin** $r \leftarrow s(L, P)$; $L^* \leftarrow L$ **end**
end {SUBLINEXT};
begin
  $L \leftarrow S \leftarrow W \leftarrow \emptyset$; {$L$, S, W are handled as queues} $r \leftarrow \infty$;

1. **for** $v \in V(D)$ **do** {vertices in $V(D)$ are in topological ordering}
   update S and W;
2. **while** $S \neq \emptyset$ **do begin**
   $\delta \leftarrow S$; augment L with δ; REMOVE(δ, D, S, W) **end**;
3. **if** D contains no arc **then begin** $r \leftarrow s(L, P)$; $L^* \leftarrow L$ **end**
   else for every π in W **do**
   SUBLINEXT(π, L, D, S, w)

end.

Theorem 1 guarantees that every strongly greedy path of D can be removed from D and added to L as a part of an optimal linear extension. Statement 2 of OPTLINEXT takes care of this step. The removal of a greedy path α from an arc diagram D is done by procedure REMOVE which, if necessary, transforms also $D - \alpha$ into a compact form and updates queues S and W. We leave implementation details of the procedure REMOVE to the reader.

The removal of strongly greedy paths from D may lead to exhausting all poset elements (arcs) of the diagram and this can happen not only for N-free posets. Otherwise, procedure OPTLINEXT attempts to build an optimal linear extension starting with an arbitrary semi-strongly greedy path π of D. This is done by the call of procedure SUBLINEXT(π, L, D, S, W) which: augments partial extension L with π, removes π from D and repeats recursively this process. To this end, procedure REMOVE is called, L is extended with strongly greedy paths of the reduced diagram and finally, if the current diagram D is non-empty, procedure SUBLINEXT is called again.

Regarding the time complexity of the algorithm note that each sequence of recursive calls of SUBLINEXT is successful in the sense that a semi-strongly greedy linear extension of the poset is obtained. Hence, it is easy to see that the time spent by procedure OPTLINEXT on producing one complete semi-strongly greedy linear extension of the given poset is bounded by $0(n + k)$, since one has only to reduce the diagram to the empty set by removing greedy paths. A poset with k dummy arcs can have at most k semi-strongly greedy paths, and the removal of a semi-strongly greedy path reduces the number of dummy arcs by at least one. Hence, there are at most $k!$ semi-strongly greedy linear extensions produced by the procedure and thus, total time complexity of the algorithm is bounded by $0(k!\,(n + k))$. This estimation is very rough since it assumes that no dummy arc is reduced from the diagram by strongly greedy paths and that exactly one dummy arc is involved with a semi-strongly greedy path.

The performance of the algorithm implemented in OPTLINEXT can be improved at least in two directions. First, it is not necessary to go forward if a partial linear extension L has already the same number of chains as L^*, the best one found so far. Secondly, the algorithm may be terminated when a linear extension is found which has the number of jumps equal to the lower bound (1).

As an illustration, let us apply the algorithm to the arc diagram D of the

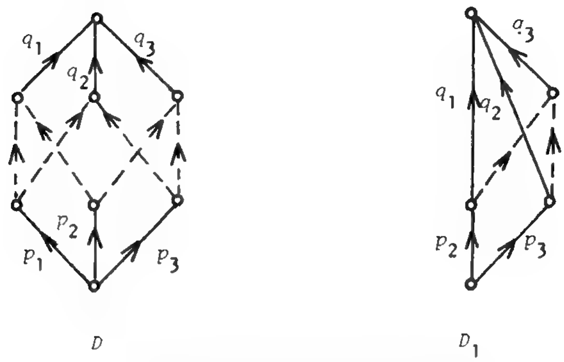

Fig. 3. Illustration of the algorithm.

3-crown which is hown in Fig. 1(c). D has no strongly greedy path and contains three semi-strongly greedy paths (p_1), (p_2), and (p_3). The diagram is symmetric with respect to each of these paths, what is not however recognized by the algorithm. We first choose the path (p_1). Fig. 3(b) shows the compact arc diagram D_1 of the poset reduced by removing (p_1). Note that this reduction results in removing four dummy arcs from the diagram. D_1 contains two strongly greedy paths (p_2, q_1) and (p_3, q_2) since q_1 and q_2 are maximal. Removing any of them we obtain a diagram which contains no dummy arcs. Therefore, the algorithm will produce only 3 complete linear extensions (instead of 6!), all with the same number of jumps. Note that in this case left-hand side of (1) can be increased by 1 since all semi-strongly greedy paths of the diagram terminate at vertices of indegree 1. Therefore, the algorithm may terminate after producing the first linear extension.

4. Conclusions

We have presented the algorithm for solving the jump number problem which is linear in the poset size and factorial in k, the number of dummy arcs in arc representations of posets. Therefore it is a linear-time algorithm in the class of posets with bounded k, which may be considered as a degree of non N-freeness of posets. The algorithm works directly on an arc diagram of a poset and searches only a portion of the solution space which consists of semi-strongly greedy linear extensions.

There exist in the literature two other algorithms for the jump number problem which are polynomial in certain restricted classes of posets. Colbourn and Pulleyblank [1] presented a dynamic programming algorithm which works on an arbitrary chain partition of a poset and finds the jump number in $0(n^m m^2)$ time, where n is the poset size and m is the number of chains in a partition. In

particular, one may apply this algorithm to a partition with the minimum number of chains which can be found efficiently by a bipartite matching algorithm.

Another algorithm, presented by Habib and Möhring [2], makes use of the substitution decomposition of posets into prime (i.e. indecomposable) posets and solves the problem in time which depends polynomially on the poset size but is a highly non-polynomial function in l, the maximum size of a prime poset. Unfortunately, the approach of Habib and Möhring requires the knowledge of the jump number of all prime posets of size at most l.

It is difficult to compare our algorithm with the two just mentioned above since there are no evident relations between the poset parameters which occur in the time complexity formulae of these algorithms. (We note only that a poset with bounded m can have an arbitrary k and conversally.) As an advantage of our approach over the two others we can list: very little preprocessing of the data, the existence of deterministic steps which depend on augmenting linear extensions with strongly greedy chains and, at last but not at least, its linear complexity in the poset size.

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LARGE SCALE NETWORK ANALYSIS WITH APPLICATIONS TO TRANSPORTATION, COMMUNICATION AND INFERENCE NETWORKS

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1. Introduction

The study of large scale networks has been mainly motivated by practical problems, like transportation problems, reliability problems. The problems usually involve finding the optimal paths in the networks and they are rather similar in nature. These different networks can be unified into a more general form of network, the semiring network. Attempts to describe such networks in a general setting is not new. In [13], Shier has described an algebraic structure to study reliability problem. Carré [6] has given an excellent description of semiring networks and their properties using matrices. In this paper, we shall describe semiring networks and show how to deal with large matrices. The latter is particularly important because in large-scale networks, even computation on the computer presents some difficulties as the amount of random access memory in every computer is limited and the computation time may be long.

2. Semirings

Let S be a non-empty set with two binary operations, $\oplus$ and $\otimes$. The system $\langle S, \oplus, \otimes \rangle$ is called a semiring if

(1) $\langle S, \oplus \rangle$ is a commutative semigroup,
(2) $\langle S, \otimes \rangle$ is a semigroup,
(3) for any $a, b, c \in S$,

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$
$$(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a).$$

For any pair of integers, n and m, let $M_{n\times m}$ be the class of all $n \times m$ matrices over the semiring $\langle S, \oplus, \otimes \rangle$. Define the operation $\oplus$ on $M_{n\times m}$ as usual: if

$A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices in $\boldsymbol{M}_{n \times m}$, let

$$A \oplus B = [a_{ij} \oplus b_{ij}], \quad \begin{array}{l} i = 1, \ldots, n, \\ j = 1, \ldots, m. \end{array}$$

Then $\langle \boldsymbol{M}_{n \times m}, \oplus \rangle$ is a commutative semigroup.

Let n, m, k be three given positive integers. Let $A = [a_{ij}]$ be a matrix in $\boldsymbol{M}_{n \times m}$, $B = [b_{ij}]$ be a matrix in $\boldsymbol{M}_{m \times k}$. Define the matrix $C = [c_{ij}]$ as follows:

$$c_{ij} = \bigoplus_{r=1,\ldots,m} (a_{ir} \otimes b_{rj}).$$

Let this operation be denoted by $C = A \otimes B$. In the case where $m = k = n$, $\langle \boldsymbol{M}_{m \times n}, \otimes \rangle$ is a semigroup. Hence the following theorem holds.

Theorem 2.1. *Let $\langle S, \oplus, \otimes \rangle$ be a semiring. For a given positive integer n, $\langle \boldsymbol{M}_{n \times n}, \oplus, \otimes \rangle$ is a semiring.*

Suppose n is a factor of two numbers, p and m, which are greater than 1. For each matrix, $A = [a_{ij}]$, in $\boldsymbol{M}_{n \times n}$, define a collection of $p \times p$ matrices A_{ij}, where $i, j = 1, \ldots, m$, as follows.

$$A_{ij} = [a_{(i-1)p+h,(j-1)p+k}], \qquad h, k = 1, \ldots, p.$$

Then the matrix $A \otimes B$ can also be obtained as specified in the following theorem.

Theorem 2.2. *Let A, B be matrices in $\boldsymbol{M}_{n \times n}$, where $n = p \times m$. Then $C = A \otimes B$ iff*

$$C_{ij} = \bigoplus_{r=1,\ldots,m} (A_{ir} \otimes B_{rj}).$$

This theorem is very useful for very large matrices because it gives us a way of decomposing the matrices.

3. Semiring networks

Let Ω denote a given finite directed graph with vertex set $V(\Omega)$ and edge set $E(\Omega)$. In this paper, we assume that $V(\Omega) = \{1, \ldots, n\}$. Suppose S_1 and S_2 are two semirings whose additive semigroups have zero elements (i.e. additive identity elements) and whose multiplicative semigroups have identity elements. Let ω be a mapping such that

$$\omega : V(\Omega) \to S_1$$

$$\omega : E(\Omega) \to S_2.$$

The function ω is called the weight function and the ordered pair $\langle \Omega, \omega \rangle$ is called a semiring network.

For a semiring network $\langle \Omega, \omega \rangle$ over the semirings $\langle S_1, \oplus, \otimes \rangle$ and $\langle S_2, \oplus, \otimes \rangle$, let $\boldsymbol{P}(\Omega)$ denote the class of all paths of the directed graphs Ω. Extend the weight function, ω to paths of Ω as follows: if $P = (v_0, v_1, \ldots, v_k)$ is a path in $\boldsymbol{P}(\Omega)$, define

$$\omega(P) = \omega(v_0, v_1) \otimes \omega(v_1, v_2) \otimes \cdots \otimes \omega(v_{k-1}, \omega_k).$$

Observe that ω maps P into an element of S_2.

In some applications, the zero elements may not exist, but this can be easily remedied. Consider an element λ that satisfies the following properties with the operations in $\langle S_2, \oplus, \otimes \rangle$:

$$\begin{aligned} &\lambda \oplus \lambda = \lambda \\ &\lambda \otimes \lambda = \lambda \\ &a \oplus \lambda = \lambda \oplus a = a \quad \text{for all } a \text{ in } S_2, \\ &a \otimes \lambda = \lambda \otimes a = \lambda \quad \text{for all } a \text{ in } S_2. \end{aligned}$$

Then $\langle S_2 \cup \{\lambda\}, \oplus, \otimes \rangle$ is also a semiring. Hence we may include λ in the set S_2 as the zero element in the semiring. Now, we are ready to define matrices for representing networks.

Let $\langle \Omega, \omega \rangle$ be a given semiring network. Suppose Ω has n vertices, $V(\Omega) = \{1, 2, \ldots, n\}$. Define its associated matrix $A = [a_{ij}]$, $i, j = 1, \ldots, n$, as follows:

$$\begin{aligned} &\text{for } i = j, \quad a_{ii} = \text{identity of } \langle S_2, \otimes \rangle, \\ &\text{for } i \neq j, \quad a_{ij} = \begin{cases} \omega(i, j) & \text{if } (i, j) \in W(\Omega), \\ \lambda & \text{otherwise.} \end{cases} \end{aligned}$$

Given an associated matrix of a semiring network, define a sequence of $n \times n$ matrices, $A_0, A_1, \ldots, A_t, \ldots$ as follows:

$$\begin{aligned} &A_0 = A \\ &A_1 = A_0 \otimes A_0 \\ &\vdots \\ &A_{t+1} = A_t \otimes A_t \\ &\vdots \end{aligned}$$

We say that the sequence $A_0, A_1, A_2, \ldots$ converges to A^* if for some non-negative integer t,

$$A^* = A_t = A_{t+1} = A_{t+2} = \cdots$$

In such a case, we call A^* the induced matrix of $\langle \Omega, \omega \rangle$.

4. Basic problems of semiring networks and their applications

Consider some basic problems of the semiring network. For any given semiring network, its induced matrix does not always exist. So the problem is, under what condition does it exist? If it exists, what are the uses of the induced matrix? Finally, how can we compute A^*?

Existence of the induced matrix A^*

It is a well known fact that the induced matrix does not necessarily exist, as testified by the shortest path problem for the transportation network that contains negative cycles. For what types of semiring networks does the induced matrix exist? In [6], Carré has studied this problem. We shall give a brief outline of the ideas presented in his paper.

In most of the applications (path problems, reliability problems, network flow problems), the additive operation $\oplus$ is usually either the minimum or maximum operation. Hence, to generalize this idea into semiring networks, we may assume that $a \oplus a = a$ and introduce an ordering (or partial ordering) to the semiring. This ordering is determined by the additive operation $\oplus$.

For a semiring, define an order relation as follows:

$$a \leq b \text{ if } a \oplus b = a,$$

and define the strict order relation as

$$a < b \text{ if } a \leq b \text{ and } a \neq b.$$

Then the following theorem gives a condition for the existence of the induced matrix.

Theorem 4.1. *If a semiring network does not contain any cycle* 0 *with* $\omega(C) < e$ (*where* e *is the multiplicative identity*), *then the induced matrix* A^* *exists.*

Uses of the induced matrix

Each (i, j) entry of the induced matrix A^* gives the value of an 'optimal' path from vertex i to vertex j. The reason why we call it the 'optimal' path may not be very obvious until we look at the applications of the induced matrix. For example, in the shortest path problem in which the edge-lengths are non-negative real numbers, the induced matrix exists and is actually the distance matrix (matrix whose entries show the distance from one vertex to another) [1, 4, 5]. Similarly, for the reliability problem described in [13], the induced matrix is the reliability matrix.

From the induced matrix, many other properties of the networks can be obtained mainly because many properties of the networks can be defined in terms of the entries of the induced matrix. Let us illustrate this by the transportation

problem with the semiring structure $\langle \boldsymbol{R} \cup \{\infty\}, \boldsymbol{L}, +\rangle$ for its edges, where : denotes the minimum operation.

The radius of a vertex, centres, centroids, total distance of a network are defined in terms of the distance of the shortest path between two vertices. the radius of a vertex k is the maximum of $a^*_{ik} + a^*_{kj}$, where i, j runs from 1 to n and it can be found as follows:

(1) from the kth row of A^*, $a^*_{k1}, a^*_{k2}, \ldots, a^*_{kn}$ find the maximum, a^*_{kj}, of all elements in the kth row.

(2) find the maximum, a^*_{ik}, of all the elements $a^*_{1k}, a^*_{2k}, \ldots, a^*_{nk}$, in the kth column of A^*.

(3) then the radius is $a^*_{ik} + a^*_{kj}$.

The centre of a network is the vertex with the smallest radius and it can be found from the induced matrix A^* as follows:

(1) Let $R = (r_1, \ldots, r_n)$ be the vector such that

$$r_1 = \text{minimum}\{a^*_{i1}, \ldots, a^*_{in}\}$$

and let $C = (c_1, \ldots, c_n)$ be the vector such that

$$c_i = \text{maximum}\{a^*_{1i}, \ldots, a^*_{ni}\}:$$

(2) let $S = R + C$; then vertex i is the centre if

$$s_i = \text{minimum}\{s_1, \ldots, s_n\}.$$

The total distance at a vertex i is defined as

$$\sum_{j=1}^{n} a^*_{ij} + \sum_{j=1}^{n} a^*_{ji},$$

and it can be obtained by adding the sum of all entries in the ith row and the sum of all entries in the ith column. The total distance of a network defined as

$$\sum_{(i,j)\in V\times V} a^*_{ij}$$

and it is equal to twice the sum of all the entries of A^*. To compare two different network designs, the total distance may be used as a form of measurement. Another measure that may be considered is the cost-effective ratio for two networks, which is defined as

$$\frac{\text{decrease in total distance}}{\text{increase in cost}},$$

and this can be obtained by computing the total distances of the networks and their respective costs.

Another application is to find a shortest path from a vertex i to another vertex j. Such a path can be obtained by the following algorithm:

(1) If $a^*_{ij} = \lambda$, then there is no path from vertex i to vertex j; stop. Otherwise, go to Step (2).

(2) Take the ith row, R_i and jth column, C_j of the induced matrix A^*.
(3) Let $S = R_i + C_j$. If there exists $k \neq i, j$ such that $s_k = a^*_{ij}$, then k is an internal vertex of a path from i to j; go to Step (4). Otherwise, (i, j) is such a path; stop.
(4) Use this algorithm to find a shortest path from i to k and a shortest path from k to j. Then these two paths constitute a shortest path from i to j.

An example in which the weight of vertices is defined is the inference network. Consider the network with vertex weight and edge weight both defined on $\langle [0, 1], \Gamma, x \rangle$, where Γ denotes the maximum operation. The inferenced value at a vertex v is

$$\max\{\omega(u) \times \omega(P)\},$$

where u runs through every vertex in $V(\Omega)$, and p runs through every path from u to v. the inferenced value of all the vertices can be found from the induced matrix A^* as follows:

(1) Let $W = (w_1, \ldots, w_n)$, where $w_i = \omega(i)$.
(2) Let $W' = W \otimes A^*$, where $\otimes$ is the matrix multiplication for matrices over the semiring $\langle [0, 1], \Gamma, x \rangle$.

Then the entries of the vector W' gives the inferenced value of each vertex in $\langle \Omega, \omega \rangle$.

The above examples do not exhaust all the applications of the induced matrix. Other types of applications occur in network flow problems, assignment problems, critical path analysis, reliability problems [2, 3, 13].

Various algorithms computing A^*

Many algorithms have evolved for computing the induced matrix. Two such algorithms are the cascade algorithm and Floyd's algorithm [5–9]. These two algorithms operate on whole matrices. Variations of these two algorithms that decompose large matrices have also been developed [10, 11]. We shall briefly describe the cascade algorithm and Floyd's algorithm below.

The Cascade Algorithm

For $i = 1$ to n
for $j = 1$ to n

$$a_{ij} = \bigoplus_{k=1,\ldots,n} (a_{ik} \otimes a_{kj})$$

Floyd's Algorithm

For $k = 1$ to n
for each pair $i, j \in \{1, \ldots, n\}$, let $c_{ij} = a_{ik} \otimes a_{kj}$ let $A = C \oplus A$ (where C denotes the matrix $[c_{ij}]$)

An algorithm using decomposition of matrices is given blow. This makes use of Theorem 2.2.

Algorithm using decomposition to compute A^*

Let $A^{[0]} = A$ and for each non-negative integer t,

$$A_{ij}^{[t+l]} = \bigoplus_{k=1,\ldots,m} (A_{ik}^{[t]} \otimes A_{kj}^{[t]}), \quad i, j \in \{1, \ldots, m\},$$

where the matrices $A_{ij}^{[t]}$, $i, j \in \{1, \ldots, m\}$, are submatrices of $A[t]$ as defined in Section 2.

5. Conclusion

In this paper, we see that the concept of semiring networks can be used to unify various network problems. The idea of imposing algebraic structure is by no means new. Other types of structures have also been discussed [13]. These trends may lead to the study of general algebraic networks.

For large networks, various methods have been devised to simplify and improve efficiency of algorithms for finding properties of networks using matrices. The availability of software that provides whole matrix operations allows us to consider algorithms which have been formerly deemed as inefficient.

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ON THE NONSEPARATING INDEPENDENT SET PROBLEM AND FEEDBACK SET PROBLEM FOR GRAPHS WITH NO VERTEX DEGREE EXCEEDING THREE

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This paper shows that both the nonseparating independent set problem and feedback set problem can be solved in polynomial time for graphs with no vertex degree exceeding 3 by reducing the problems to the matroid parity problem.

1. Introduction

Let $G = (V, E)$ be a graph where V and E are the sets of vertices and edges of G, respectively. For $S \subset V$, $G - S$ is the graph obtained from G by deleting the vertices in S.

A set of vertices I is said to be independent if no two vertices in I are joined by an edge. The independent set problem is to find a maximum independent set in G. It is well-known that this problem is NP-hard even for planar graphs with no vertex degree exceeding 3 [1].

A set of vertices X is called a separating set if the number of connected components of $G - X$ is more than that of G. The nonseparating independent set problem is to find a maximum independent set which contains no separating set. Note that this is equivalent to the connected dominating set problem. This problem is also NP-hard even for planar graphs with no vertex degree exceeding 4 [2].

A set of vertices F is called a feedback set of G if $G - F$ contains no circuit. The feedback set problem which is to find a minimum feedback set is known to be NP-hard [3].

This paper shows that both of the nonseparating independent set problem and feedback set problem can be solved in polynomial time for graphs with no vertex degree exceeding 3. We will indicate that the nonseparating independent set problem and the feedback set problem of a 3-regular graph are closely related in the sense that they correspond to the matching problem and the spanning set problem of some linearly represented 2-polymatoid, respectively.

2. Nonseparating independent set problem

2.1. *Reduction to the problem for* 3*-regular graphs*

A set $I \subset V$ is called a nonseparating independent set if I satisfies the following two conditions:

(N1) I is an independent set

(N2) I contains no separating set.

We call I a maximum nonseparating independent set if I contains maximum number of vertices. $\nu(G)$ is the number of vertices in a maximum nonseparating independent set of G. $d_G(v)$ is the degree of vertex v in G, that is, the number of edges incident to v in G. $G(A)$ is the induced subgtaph on $A \subset V$, that is, $G(A) = G - (V - A)$.

We assume that G is a multigraph with no vertex degree exceeding 3. We allow parallel edges and loops. Note that an independent set contains no endvertex of a loop by definition.

Let $G(X)$ be a connected component of G such that $|X| \leqslant 2$. It is easy to see that $\nu(G) = \nu(G - X) + 1$ if at least one vertex of X has no loop, and $\nu(G) = \nu(G - X)$ otherwise. Hence we may assume that each connected component of G has at least three vertices.

Let v be a vertex of degree one, and H be the graph obtained from G by adding two vertices x and y, connecting them by two parallel edges, and adding edges (v, x) and (v, y). Note that $d_H(v) = d_H(x) = d_H(y) = 3$. By the assumption that each connected component of G has at least three vertices, the vertex adjacent to v is a cut-vertex of G and H, and hence we have $\nu(G) = \nu(H)$.

Let v be a vertex of degree two, and u and w be the vertices adjacent to v. u and w may be identical. Let H be the graph obtained from G by deleting v, adding two vertices x and y, connecting them by two parallel edges, and adding edges (x, u) and (y, w). Note that $d_H(x) = d_H(y) = 3$. It is easy to see that $\nu(G) = \nu(H)$.

In either case, we can easily construct a maximum nonseparating independent set of G from that of H. Since we can apply the above operations until there exists no vertex of degree less than 3, the nonseparating independent set problem for graphs with no vertex degree exceeding 3 is reduced to the problem for 3-regular graphs.

2.2. *Reduction to the matching problem of* 2*-polymatroids*

We assume that $G = (V, E)$ is a 3-regular graph. $\mu(G)$ is the nullity (cyclomatic number) of G. Define for each $X \subset V$

$$f(X) = \mu(G) - \mu(G - X).$$

It is merely a routine work to verify that function f satisfies the following

conditions:

(P1) $f(\emptyset)=0$

(P2) $f(X) \leqslant f(Y)$ if $X \subset Y$

(P3) $f(X \cup Y)+f(X \cap Y) \leqslant f(X)+f(Y)$ for any $X, Y \subset V$

(P4) $f(\{x\}) \leqslant 2$ for each $x \in V$.

Thus $P(G)=(V, f)$ is a 2-polymatroid (this is called a ($\leqslant$2)-polymatroid by Lovász [4]). Note that (P1)–(P3) hold for any graph while (P4) comes from the assumption that G is 3-regular ($f(\{x\}) \leqslant d_G(x)-1$ in general). Note also that $f(\{x\})<2$ if and only if x is a cut-vertex or the endvertex of a loop. $X \subset V$ is called a matching of $P(G)$ if $f(X)=2|X|$.

Theorem 1. *$X \subset V$ is a matching of $P(G)$ if and only if X is a nonseparating independent set of G.*

Proof. We first show that if X is a nonseparating independent set of G, X is a matching of $P(G)$. We shall apply induction on $|X|$. The theorem is trivially true if $|X|=1$. For any $v \in X$, $X-\{v\}$ is a nonseparating independent set of G. Since $|X-\{v\}|<|X|$, $f(X-\{v\})=2|X-\{v\}|$ by the induction hypothesis. $d_{G-(X-v)}(v)=3$ and v is not a cut-vertex of $G-(X-\{v\})$ or the endvertex of a loop, for X is a nonseparating independent set of G. Hence $f(X)=f(X-\{v\})+2=2|X|$.

Now, we shall show that if X is not a nonseparating independent set of G, $f(X)<2|X|$. We may assume that X is a minimal one by (P3). If X consists of one vertex v, v is a cut-vertex of G or the endvertex of a loop, and hence $f(X)<2|X|=2$. If X consists of more than one vertex, $X-\{v\}$ is a nonseparating independent set of G for any $v \in X$. Since X is not a nonseparating independent set of G, v is adjacent to some vertex in $X-\{v\}$ or there exists a separating set $A \subset X$ containing v. Hence, $d_{G-(X-v)}(v) \leqslant 2$ or v is a cut-vertex of $G-(X-\{v\})$, and so $f(X) \leqslant f(X-\{v\})+1<2|X|$. □

Lovász presented a polynomial-time algorithm to obtain a maximum matching of a linearly represented 2-polymatroid (which will be defined later) [4]. Hence, by Theorem 1, we can solve the nonseparating independent set problem for G in polynomial time provided that we can obtain a linear representation of $P(G)$ in polynomial time.

We shall show that there exists a linear representation of $P(G)$ for any G.

2.3. *Linear representation of $P(G)$*

Let $M^d(G)=(E, \rho^d)$ be the dual matroid of the circuit matroid of $G=(V, E)$, where ρ^d is the rank function of $M^d(G)$.

Construct $M^*(G)=(E^*, \rho^*)$ from $M^d(G)$ and G as follows:

(1) For each $v \in V$, add two elements $e_1^*(v)$ and $e_2^*(v)$ to $M^d(G)$ so that e_1^* and

$e_2^*(v)$ are parallel to arbitrary distinct elements of $\Delta_G(v)$, where $\Delta_G(v)$ is the set of edges incident to v in G.

(2) Delete E from the resulting matroid.

Since $M^d(G)$ is a linearly representable matroid, so is $M^*(G)$. We can obtain in polynomial time a linear representation of $M^*(G)$ from graph G represented by, for example, the incidence matrix.

Given a matroid (E, r) and subsets $A_1, A_2, \ldots, A_n \subset E$, we can define a polymatroid (S, r') on the set $S = \{A_1, A_2, \ldots, A_n\}$ by

$$r'(X) = r(\bigcup \{A \mid A \in X\}) \qquad (X \subset S).$$

If (E, r) is linearly represented, (S, r') is said to be linearly represented and we call (E, r) a linear representation of (S, r').

Theorem 2. *$M^*(G)$ is a linear representation of $P(G) = (V, f)$.*

Proof. Let ρ be the rank function of G. Define for any $X \subset V$, $\Delta(X) \subset E$ and $X^* \subset E^*$ as follows:

$$\Delta(X) = \{e \mid e \in E, e \text{ is incident to some } v \in X \subset V\}$$
$$X^* = \{e_i^*(v) \mid v \in X \subset V, i = 1, 2\}.$$

By the construction of $M^*(G)$,

$$\begin{aligned}\rho^*(X^*) = \rho^d(\Delta(X)) &= |\Delta(X)| + \rho(E - \Delta(X)) - \rho(E)\\ &= |\Delta(X)| + \rho(E - \Delta(X)) + \mu(G) - |E|\\ &= \mu(G) - (|E| - |\Delta(X)| - \rho(E - \Delta(X)))\\ &= \mu(G) - (|E - \Delta(X)| - \rho(E - \Delta(X)))\\ &= \mu(G) - \mu(G - X)\\ &= f(X).\end{aligned}$$

Hence $P(G)$ is a 2-polymatroid on the set $\{\{e_1^*(v), e_2^*(v)\} \mid v \in V\}$. $\square$

Thus, the nonseparating independent set problem can be solved in polynomial time for graphs with no vertex degree exceeding 3.

3. Feedback set problem

3.1. *Reduction to the problem for 3-regular graphs*

A set $F \subset V$ is called a feedback set if $G - F$ contains no circuit. We call F a minimum feedback set if F contains a minimum number of vertices. $\eta(G)$ is the number of vertices in a minimum feedback set of G.

We assume as in Section 2 that G is a multigraph with no vertex degree exceeding 3.

Let v be a vertex of degree one. It is trivial that $\eta(G) = \eta(G - \{v\})$.

Let v be a vertex of degree two which is the endvertex of a loop. Then $\eta(G) = \eta(G - \{v\}) + 1$.

Hence we may assume that G has neither a vertex of degree one nor a vertex of degree two which is the endvertex of a loop.

Let v be a vertex of degree two, and u and w be the vertices adjacent to v. u and w may be identical. Let H be the graph obtained from G by deleting v and adding an edge (u, w). It is easy to see that $\eta(G) = \eta(H)$, and we can easily construct a minimum feedback set of G from that of H.

Since we can apply the above operations until there exists no vertex of degree less than 3, the feedback set problem for graphs with no vertex degree exceeding 3 is reduced to the problem for 3-regular graphs.

3.2. *Relation to the nonseparating independent set problem*

A set $T \subset S$ of a 2-polymatroid $P = (S, r)$ is called a spanning set of P if $r(T) = r(S)$. The following equality which is a generalization of Gallai's identity for graphs is proved in [4].

$$|T| + |D| = r(S),$$

where T and D are a minimum spanning set and a maximum matching of P, respectively.

We assume that G is a 3-regular graph. From the definitions of the spanning set, the 2-polymatroid $P(G)$, and the feedback set, we have the following theorem.

Theorem 3. *$X \subset V$ is a spanning set of $P(G)$ if and only if X is a feedback set of G.*

By Theorems 1 and 3, and the generalized Gallai's identity, we have the following theorem, which was proved for connected cubic graphs by Speckenmeyer [5].

Theorem 4. *For a 3-regular graph $G = (V, E)$,*

$$\eta(G) + \nu(G) = \mu(G).$$

By this theorem and the fact that for any maximum nonseparating independent set I of G, each 2-connected component of $G - I$ has at most one circuit, we can easily construct a minimum feedback set from I. Since we can obtain a maximum nonseparating independent set in polynomial time as shown in the previous section, we can solve the feedback set problem in polynomial time for graphs with no vertex degree exceeding 3.

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P_3-FACTORIZATION OF COMPLETE BIPARTITE GRAPHS

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In this paper, it is shown that a necessary and sufficient condition for the existence of a P_3-factorization of $K_{m,n}$ is (i) $m+n \equiv 0 \pmod 3$, (ii) $m \leqslant 2n$, (iii) $n \leqslant 2m$ and (iv) $3mn/2(m+n)$ is an integer.

1. Introduction

Let P_3 be a *path* on 3 points and $K_{m,n}$ be a *complete bipartite graph* with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$. A spanning subgraph F of $K_{m,n}$ is called a *P_3-factor* if each component of F is isomorphic to P_3. If $K_{m,n}$ is expressed as a line-disjoint sum of P_3-factors, then this sum is called a *P_3-factorization* of $K_{m,n}$.

In this paper, a necessary and sufficient condition for the existence of a P_3-factorization of $K_{m,n}$ will be given.

2. P_3-factor of $K_{m,n}$

The following theorem is on the existence of P_3-factors of $K_{m,n}$.

Theorem 1. *$K_{m,n}$ has a P_3-factor if and only if* (i) $m+n \equiv 0$ (*mod* 3), (ii) $m \leqslant 2n$ *and* (iii) $n \leqslant 2m$.

Proof. Suppose that $K_{m,n}$ has a P_3-factor F. Let t be the number of components of F. Then $t = \frac{1}{3}(m-n)$. Hence, Condition (i) is necessary. Among these t components, let x and y be the number of components whose endpoints are in V_2 and V_1, respectively. Then, since F is a spanning subgraph of $K_{m,n}$, we have $x+2y=m$ and $2x+y=n$. Hence $x=\frac{1}{3}(2n-m)$ and $y=\frac{1}{3}(2m-n)$. From $0 \leqslant x \leqslant m$ and $0 \leqslant y \leqslant n$, we must have $m \leqslant 2n$ and $n \leqslant 2m$. Conditions (ii) and (iii) are, therefore, necessary.

For those parameters m and n satisfying (i)–(iii), let $x=\frac{1}{3}(2n-m)$ and $y=\frac{1}{3}(2m-n)$. Then x and y are integers such that $0 \leqslant x \leqslant m$ and $0 \leqslant y \leqslant n$. Hence $x+2y=m$ and $2x+y=n$. Using x points in V_1 and $2x$ points in V_2, consider x P_3's whose endpoints are in V_2. Using the remaining $2y$ points in V_1 and the

remaining y points in V_2, consider y P_3's whose endpoints are in V_1. Then these $x+y$ P_3's are line-disjoint and they form a P_3-factor of $K_{m,n}$. □

Corollary 1. *$K_{n,n}$ has a P_3-factor if and only if $n \equiv 0 \pmod 3$.*

3. P_3-factorization of $K_{m,n}$

Our main theorem is on the existence of P_3-factorizations of $K_{m,n}$.

Theorem 2. *$K_{m,n}$ has a P_3-factorization if and only if* (i) $m+n \equiv 0$ (*mod* 3), (ii) $m \leqslant 2n$, (iii) $n \leqslant 2m$ *and* (iv) $3mn/2(m+n)$ *is an integer.*

Proof. Suppose that $K_{m,n}$ is factorized into r P_3-factors. By Theorem 1, Conditions (i)–(iii) are obviously necessary. Let t be the number of components of each P_3-factors. Then $t = \frac{1}{3}(m+n)$ and $r = 3mn/2(m+n)$. Hence, condition (iv) is necessary. The proof of sufficiency will be given in Subsection 3.2.

3.1. *Extension theorem of P_3-factorization of $K_{m,n}$*

We prove the following extension theorem, which we use later in the paper.

Theorem 3. *If $K_{m,n}$ has a P_3-factorization, then $K_{sm,sn}$ has a P_3-factorization for every positive integer s.*

Proof. Let V_1, V_2 be the independent sets of $K_{sm,sn}$ where $|V_1| = sm$ and $|V_2| = sn$. Divide V_1 and V_2 into s subsets of m and n points each, respectively. Construct a new graph G with a point set consisting of the subsets which were just constructed. In this graph, two points are adjacent if and only if the subsets come from disjoint independent sets of $K_{sm,sn}$. G is a complete bipartite graph $K_{s,s}$. Noting that the cardinality of each subset identified with a point set of G is m or n and that $K_{s,s}$ has a 1-factorization, we see that the desired result is obtained. 1-factorizations of $K_{s,s}$ are discussed in [1, 2]. □

3.2. *The proof of the sufficiency of Theorem* 2

There are three cases to consider.

Case (1) $m = 2n$: In this case, from Theorem 3, $K_{2n,n}$ has a P_3-factorization since $K_{2,1}$ is just P_3.

Case (2) $n = 2m$: Obviously, $K_{m,2m}$ has a P_3-factorization.

Case (3) $m < 2n$ and $n < 2m$: In this case, let $x = \frac{1}{3}(2n-m)$, $y = \frac{1}{3}(2m-n)$, $t = \frac{1}{3}(m+n)$ and $r = 3mn/2(m+n)$. Then from Conditions (i)–(iv), x, y, t, r

are integers and $0<x<m$ and $0<y<n$. We have $x+2y=m$ and $2x+y=n$. Hence $r=(x+y)+xy/2(x+y)$. Let $z=xy/2(x+y)$, which is a positive integer. And let $(x, 2y)=d$, $x=dp$, $2y=dq$, where $(p, q)=1$. Then dp is even and $z=dpq/2(2p+q)$. The following lemmas can be verified.

Lemma 1. $(p, q)=1 \Rightarrow (pq, p+q)=1$.

Lemma 2. $(p, q)=1 \Rightarrow (pq, 2p+q)=1$ *(q: odd) or* 2 *(q: even)*.

Using these p, q, d the parameters m and n satisfying Conditions (i)–(iv) are expressed as follows:

Lemma 3. $(p, q)=1$ *and* $dpq/2(2p+q)$ *is an integer*

$\Rightarrow$
(I) $m=2(p+q)(2p+q)s$, $n=(4p+q)(2p+q)s$ *when q is odd,*
(II) $m=2(p+2q')(p+q')s$, $n=2(2p+q')(p+q')s$ *when* $q=2q'$ *and* q' *is odd,*
(III) $m=(p+4q'')(p+2q'')s$, $n=2(p+q'')(p+2q'')s$ *when* $q=4q''$, *where s is a positive integer.*

We use the following notations for sequences.

Notation. Let A and B be two sequences of the same size such as
A: $a_1, a_2, \ldots, a_u$
B: $b_1, b_2, \ldots, b_u$.
If $b_i=a_i+c$ $(i=1, 2, \ldots, u)$, then we write $B=A+c$. If $b_i=((a_i+c) \bmod w)$ $(i=1, 2, \ldots, u)$, then we write $B=A+c \bmod w$, where the residuals $a_i+c \bmod w$ are integers in the set $\{1, 2, \ldots, w\}$.

For the parameters m and n in (I)–(III) when $s=1$, we can construct a P_3-factorization of $K_{m,n}$.

Lemma 4. $(p, q)=1$ *and q is odd*

$$m=2(p+q)(2p+q),\ n=(4p+q)(2p+q)$$

$\Rightarrow$ *$K_{m,n}$ has a P_3-factorization.*

Proof. The proof is by construction (Algorithm I). Let $x=\frac{1}{3}(2n-m)$, $y=\frac{1}{3}(2m-n)$, $t=\frac{1}{3}(m+n)$, $r=3mn/2(m+n)$. Then we have $x=2p(2p+q)$, $y=q(2p+q)$, $t=(2p+q)^2$, $r=(p+q)(4p+q)$. Let $r_1=p+q$, $r_2=4p+q$, $m_0=m/r_1=2(2p+q)$, $n_0=n/r_2=2p+q$. Consider two sequences R and C of the same size $4(2p+q)$.

R: $1, 1, 2, 2, \ldots, 2(2p+q), 2(2p+q)$

C: $1, 2, \ldots, 4(2p+q)-1, 4(2p+q)$.

Construct p sequences R_i such that $R_i = R + 2(i-1)(2p+q)$ $(i = 1, 2, \ldots, p)$. Construct p sequences C_i such that $C_i = (C + 2(i-1) \bmod 4(2p+q)) + 4(i-1)(2p+q)$ $(i = 1, 2, \ldots, p)$. Consider two sequences R' and C' of the same size $2(2p+q)$.

R': $1, 2, \ldots, 2(2p+q)-1, 2(2p+q)$

C': $1, 3, \ldots, 2p+q, 2, 4, \ldots, 2p+q-1, 1, 3, \ldots, 2p+q, 2, 4, \ldots, 2p+q-1$.

Construct q sequences R_i' such that $R_i' = R' + 2(i-1)(2p+q) + 2p(2p+q)$ $(i = 1, 2, \ldots, q)$. Construct q sequences C_i' such that $C_i' = (C' + (i-1) + 2p \bmod 2p+q) + (i-1)(2p+q) + 4p(2p+q)$ $(i = 1, 2, \ldots, q)$. Consider two sequences I and J of the same size.

I: $R_1, R_2, \ldots, R_p, R_1', R_2', \ldots, R_q'$

J: $C_1, C_2, \ldots, C_p, C_1', C_2', \ldots, C_q'$.

Then the size of I or J is $2t$. Let i_k and j_k be the kth element of I and J, respectively $(k = 1, 2, \ldots, 2t)$. Join two points i_k in V_1 and j_k in V_2 with a line (i_k, j_k) $(k = 1, 2, \ldots, 2t)$. Construct a graph F with two point sets $\{i_k\}$ and $\{j_k\}$ and a line set $\{(i_k, j_k)\}$. Then F is a P_3-factor of $K_{m,n}$. This graph is called a *P_3-factor constructed with two sequences I and J.*
Construct r_1 sequences I_i such that $I_i = I + (i-1)m_0 \bmod m$ $(i = 1, 2, \ldots, r_1)$. Construct r_2 sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ $(j = 1, 2, \ldots, r_2)$. Construct $r_1 r_2$ P_3-factors F_{ij} with I_i and J_j $(i = 1, 2, \ldots, r_1; j = 1, 2, \ldots, r_2)$. Then it is easy to show that F_{ij} are line-disjoint and that their sum is a P_3-factorization of $K_{m,n}$. $\square$

Lemma 5. *$(p, q) = 1$ and $q = 2q'$ (q': odd)*

$$m = 2(p+2q')(p+q'),\ n = 2(2p+q')(p+q')$$

$\Rightarrow K_{m,n}$ has a P_3-factorization.

Proof. The proof is by construction (Algorithm II). Let $x = \frac{1}{3}(2n-m)$, $y = \frac{1}{3}(2m-n)$, $t = \frac{1}{3}(m+n)$, $r = 3mn/2(m+n)$. Then we have $x = 2p(p+q')$, $y = 2q'(p+q')$, $t = 2(p+q')^2$, $r = (p+2q')(2p+q')$. Let $r_1 = p+2q'$, $r_2 = 2p+q'$, $m_0 = m/r_1 = 2(p+q')$, $n_0 = n/r_2 = 2(p+q')$. Consider two sequences R and C of the same size $4(p+q')$.

R: $1, 1, 2, 2, \ldots, 2(p+q'), 2(p+q')$

C: $1, 2, \ldots, 4(p+q')-1, 4(p+q')$.

Construct p sequences R_i such that $R_i = R + 2(i-1)(p+q')$ $(i = 1, 2, \ldots, p)$. Construct p sequences C_i such that $C_i = (C + 2(i-1) \bmod 4(p+q')) + 4(i-

$1)(p+q')$ $(i=1,2,\ldots,p)$. Consider two sequences R' and C' of the same size $4(p+q')$.

R': $1, 2, \ldots, 4(p+q')-1, 4(p+q')$

C': $1, 3, \ldots, 2(p+q')-1, 1, 3, \ldots, 2(p+q')-1, 2, 4, \ldots, 2(p+q'),$
$2, 4, \ldots, 2(p+q')$.

Construct q' sequences R_i' such that $R_i' = R' + 4(i-1)(p+q') + 2p(p+q')$ $(i=1,2,\ldots,q')$. Construct q' sequences C_i' such that $C_i' = (C' + 2(i-1) + 2p \bmod 2(p+q')) + 2(i-1)(p+q') + 4p(p+q')$ $(i=1,2,\ldots,q')$.
Consider two sequences I and J of the same size $2t$.

I: $R_1, R_2, \ldots, R_p, R_1', R_2', \ldots, R_{q'}'$.

J: $C_1, C_2, \ldots, C_p, C_1', C_2', \ldots, C_{q'}'$.

Construct r_1 sequences I_i such that $I_i = I + (i-1)m_0 \bmod m$ $(i=1,2,\ldots,r_1)$. Construct r_2 sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ $(j=1,2,\ldots,r_2)$. Construct r_1r_2 P_3-factors F_{ij} with I_i and J_j $(i=1,2,\ldots,r_1; j=1,2,\ldots,r_2)$. Then it is easy to show that F_{ij} are line-disjoint and that their sum is a P_3-factorization of $K_{m,n}$. □

Lemma 6. $(p,q)=1$ *and* $q=4q''$

$$m=(p+4q'')(p+2q''),\ n=2(p+q'')(p+2q'')$$

$\Rightarrow$ $K_{m,n}$ *has a* P_3*-factorization.*

Proof. The proof is by construction (Algorithm III). Let $x=\frac{1}{3}(2n-m)$, $y=\frac{1}{3}(2m-n)$, $t=\frac{1}{3}(m+n)$, $r=3mn/2(m+n)$. Then we have $x=p(p+2q'')$, $y=2q''(p+2q'')$, $t=(p+2q'')^2$, $r=(p+4q'')(p+q'')$. Let $r_1=p+4q''$, $r_2=p+q''$, $m_0=m/r_1=p+2q''$, $n_0=n/r_2=2(p+2q'')$. Consider two sequences R and C of the same size $2(p+2q'')$.

R: $1, 1, 2, 2, \ldots, p+2q'', p+2q''$

C: $1, 2, \ldots, 2(p+2q'')-1, 2(p+2q'')$.

Construct p sequences R_i such that $R_i = R + (i-1)(p+2q'')$ $(i=1,2,\ldots,p)$. Construct p sequences C_i such that $C_i = (C + 2(i-1) \bmod 2(p+2q'')) + 2(i-1)(p+2p'')$ $(i=1,2,\ldots,p)$. Consider two sequences R' and C' of the same size $4(p+2q'')$.

R': $1, 2, \ldots, 4(p+2q'')-1, 4(p+2q'')$

C': $1, 3, \ldots, 2(p+2q'')-1, 2, 4, \ldots, 2(p+2q''), 3, 5, \ldots, 2(p+2q'')-1, 1,$
$4, 6, \ldots, 2(p+2q''), 2$.

Construct q'' sequences R_i' such that $R_i' = R' + 4(i-1)(p+2q'') + p(p+2q'')$ $(i=1,2,\ldots,q'')$. Construct q'' sequences C_i' such that $C_i' = (C' + 4(i-1) +$

$2p \bmod 2(p+2q'')) + 2(i-1)(p+2q'') + 2p(p+2q'')$ $(i = 1, 2, \ldots, q'')$. Consider two sequences I and J of the same size $2t$.

I: $R_1, R_2, \ldots, R_p, R'_1, R'_2, \ldots, R'_{q''}$

J: $C_1, C_2, \ldots, C_p, C'_1, C'_2, \ldots, C'_{q''}$.

Construct r_1 sequences I_i such that $I_i = I + (i-1)m_0 \bmod m$ $(i = 1, 2, \ldots, r_1)$. Construct r_2 sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ $(j = 1, 2, \ldots, r_2)$. Construct $r_1 r_2$ P_3-factors F_{ij} with I_i and J_j $(i = 1, 2, \ldots, r_1; j = 1, 2, \ldots, r_2)$. Then it is easy to show that F_{ij} are line-disjoint and that their sum is a P_3-factorization of $K_{m,n}$. □

Applying Theorem 3 with Lemmas 4 to 6, it can be seen that for the parameters m and n satisfying Conditions (i)–(iv), $K_{m,n}$ has a P_3-factorization. This completes the proof of Theorem 2.

Corollary 2. *$K_{n,n}$ has a P_3-factorization if and only if $n \equiv 0$ (mod 12).*

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n-GRAPHS

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During the past few years papers have appeared that take a graph theoretic approach to the investigation of PL-manifolds. These graphs have variously been called crystallizations, gems (graph encoded manifolds) and n-graphs. The basic ideas and major results of this combinatorial method are surveyed in this paper.

1. Introduction

The use of combinatorial methods in topology is certainly not new; simplicial homology is an obvious example. However the graph theoretic approach to be surveyed here is fairly recent, due to Pezzana, Ferri, Gagliardi, Lins, Mandel, Vince and others. This paper is not intended as a complete review of the subject; for this we refer to [7] where an extended list of references can also be found. Instead we present only major results with motivation. Proofs are either sketched or omitted. A reference is given for each result but the proof sketched may not be the original.

Graphs can have multiple edges and $V(G)$ and $E(G)$ denote the point and edge sets of G respectively. Let $[n]$ denote the set $\{0, 1, \ldots, n-1\}$. An ***n*-graph** is a graph G, regular of degree n, together with an edge coloring $E(G) \rightarrow [n]$ such that incident edges are different colors. The motivation for this definition is that an $(n+1)$-graph G encodes an n-dimensional simplicial complex ΔG as follows. For each point v of G let σ_v be an n-simplex whose set of $n+1$ vertices is in bijection with $[n+1]$. Let k be the disjoint union of the σ_v, $v \in V(G)$. For each $i \in [n+1]$ identify the $(n-1)$-face of σ_u colored $[n+1]-\{i\}$ with the $(n-1)$-face of σ_v colored $[n+1]-\{i\}$ if and only if u and v are joined in G by an edge colored i. If $\sim$ is this indentification then $K/\sim$ is denoted ΔG and the underlying topological space of ΔG is denoted $|G|$. Fig. 1 is an example of a 3-graph G and the corresponding 2-dimensional complex ΔG. The underlying space $|G|$ is the 2-sphere.

Throughout this paper manifolds will be PL, compact, and without boundary. The simple, but basic, fact in the theory is that any PL n-manifold M is homeomorphic to $|G|$ for some $(n+1)$-graph G [2, 19, 21]. Such a G is obtained as the appropriately colored dual 1-skeleton of the barycentric subdivision of any triangulation of M. Recall that the dual 1-skeleton of a triangulation is the graph

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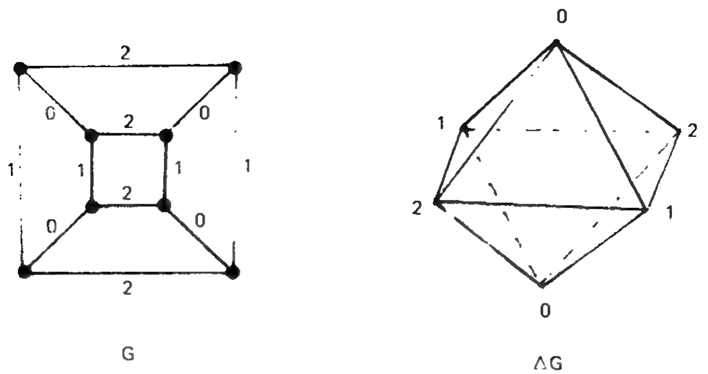

Fig. 1. A 3-graph and the associated simplicial complex ΔG.

whose points are the facets and two points are joined by an edge if and only if the corresponding n-simplices share a codimension 1 simplex. Hereafter $\approx$ denotes homeomorphism.

Theorem 1. *For any PL n-manifold M there exists an $(n+1)$-graph G such that $|G| \approx M$.*

It is known that every 1, 2, or 3-manifold can be triangulated and hence can be encoded as a 2, 3, or 4-graph. Therefore in these low dimensions the scheme is completely general. In Fig. 2, for example, are graphs representing the sphere product $S^1 \times S^2$ and the non-orientable sphere bundle $S^1 \underset{\sim}{\times} S^2$, which will be useful later. *Our point of view from here on is to regard an n-manifold as an*

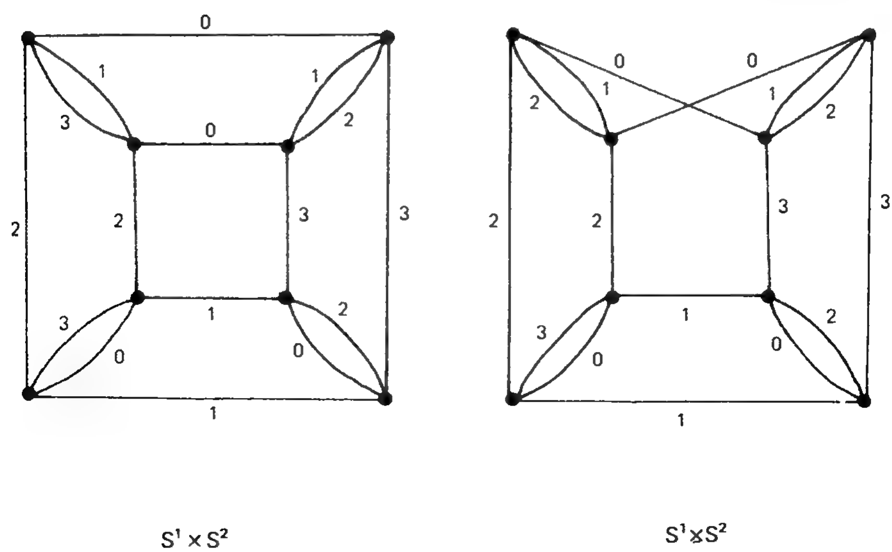

Fig. 2. Encodings of the orientable and non-orientable 2-sphere bundles over S^1.

$(n+1)$-graph. The goal is to gain topological insight into the space $|G|$ from the combinatorics of the graph G.

The basic notions about n-graphs addressed in this paper are (1) fusion, (2) canonical forms, (3) fundamental group and (4) regular embedding. Basic results connecting graphs and manifolds are contained in Section 2. In Section 3 equivalence of n-graphs is defined. Equivalence of graphs G_1 and G_2 corresponds to homeomorphism between the topological spaces $|G_1|$ and $|G_2|$. An equivalence step is one of several types of fusion, a basic operation on n-graphs that is also discussed in Section 3. The ideal situation would be to have, for each PL manifold, a unique canonical n-graph. Unfortunately this is known only for $n \leqslant 3$ and is covered in Section 4. Two simple algorithms on an n-graph G are given in Section 5 for determining the fundamental group of $|G|$. Regular embedding of an n-graph on a closed surface provides a new topological invariant for manifolds—the graph theoretic genus. For 2-manifolds this invariant is the ordinary genus of the surface, and for 3-manifolds it is essentially the Heegaard genus. This and other properties of embedding are discussed in Section 6.

2. Graph encoded manifolds

Although proofs concerning n-graphs are often combinatorial, the next result provides the link between the combinatorics of the graph G and the topology of the complex ΔG. For an n-graph G and for $J \subset [n]$ let G_J denote the subgraph of G obtained by deleting all edges with colors not in J. Each connected component of G_J is a $|J|$-graph and is called a *residue of type* J or $|J|$-*residue*. In particular, G itself is the only n-residue; each point of G is a 0-residue; each edge is a 1-residue; 2-colored cycles in G are 2-residues, etc.

Theorem 2. *There is a 1–1, inclusion reversing correspondence between the residues of an $(n+1)$–graph G and the simplices of ΔG. For $i \in [n]$, i-residues in G correspond to $(n-i)$-simplices in ΔG.*

By the above theorem, facets (highest dimensional simplices) of ΔG correspond to points of G; codimension 1-faces of ΔG corresponds to edges of G; . . . ; vertices of ΔG correspond to n-residues of G. Hence links of vertices in ΔG are encoded by n-residues; in general, links of i-faces in ΔG are encoded by $(n-i)$-residues of G.

A 2-graph encodes the circle S^1; a 3-graph encodes a surface. For an n-graph, $n \geqslant 3$, the underlying space $|G|$ is not necessarily a manifold because the link of a vertex in ΔG is not necessarily a sphere. As a consequence of Theorem 2 a 4-graph encodes a 3-manifold exactly if each 3-residue encodes a 2-sphere. Moreover, application of the Euler characteristic to 3-residues results in the following necessary and sufficient condition for a 4-graph to encode a manifold [2, 19].

Theorem 3. *Let v denote the number of vertices in a 4-graph G and r_2, r_3 the number of 2 and 3-residues, resp. Then $v \geqslant r_2 - r_3$ with equality if and only if $|G|$ is a manifold.*

A space $|G|$ is orientable if there is a coherent orientation of the facets in ΔG. In terms of the graph G this is expressed as follows [21].

Theorem 4. *The topological space $|G|$ is orientable if and only if the graph G is bipartite.*

3. Equivalent n-graphs

A basic construction in the theory of n-graphs is fusion. Consider two points u and v in an n-graph G(or in two n-graphs G_1 and G_2) and let G^* be the n-graph obtained by

(1) removing u, v and all edges connecting them;

(2) reconnecting the 'free' edges (previously incident to one of u or v) of like color.

Then G^* is said to be obtained from G by *fusion* on u and v. If there are $m \geqslant 1$ edges connecting u and v then the graph removed in step (1) is called an *m-dipole* and the fusion is called *removing a dipole*. The inverse operation is called *adding a dipole*. If J denotes the set of colors of a dipole D of an n-graph G and if u and v lie in the same residue of type $[n]-J$, then D is called *degenerate*. Otherwise D is *non-degenerate*.

Call two n-graphs *equivalent* if one can be obtained from the other by a sequence of adding or removing non-degenerate dipoles. Fig. 3 shows three equivalent 3-graphs. First dipole d_1 is added and then dipole d_2 is removed. Removing a non-degenerate dipole in G corresponds in $|G|$ to removing a ball and identifying two hemispheres on the boundary in the natural way. This makes the 'if' part of the following theorem reasonable. What is surprising is that the converse is also true [4].

Theorem 5. $|G_1| \simeq |G_2|$ *if and only if G_1 and G_2 are equivalent.*

The graph of Fig. 3c encodes the Klein bottle (see Section 4). Hence, by Theorem 5, Fig. 3a also encodes the Klein bottle.

The topological consequences of other types of fusion are summarized in the following theorem. If points u and v lie in distinct n-graphs G_1 and G_2 then fusion is denoted by $G_1 \overset{uv}{\#} G_2$. Recall that the connected sum $M_1 \# M_2$ of two manifolds is obtained by removing an open ball from each and identifying the two spherical boundaries via a homeomorphism. Also $\otimes$ will stand for the orientable (ordinary product) or non-orientable bundle. Parts (a, b, c, d) of the theorem are found in [4, 17, 12, 8], respectively.

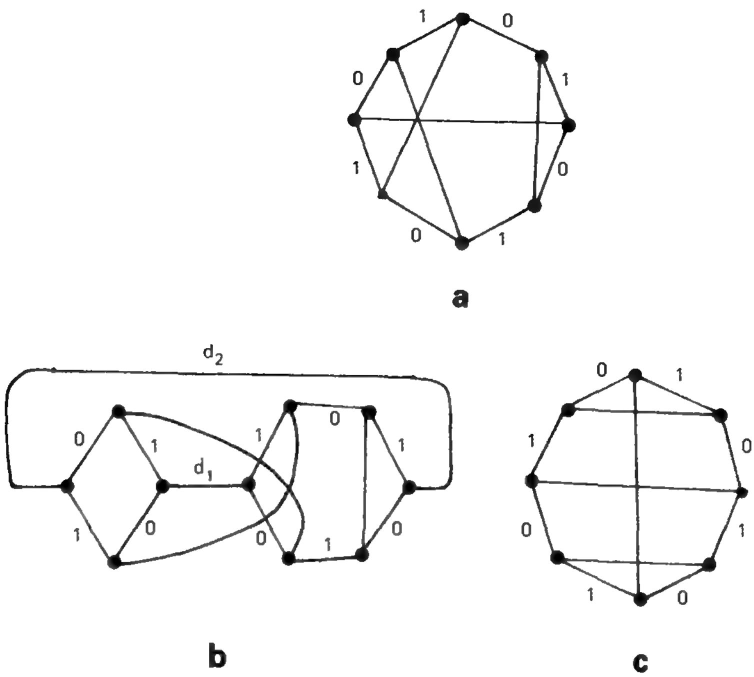

Fig. 3. Equivalent 3-graphs.

Theorem 6. *For 4-graphs that encode 3-manifolds*

(a) $|G_1 \overset{uv}{\#} G_2| \simeq |G_1| \# |G_2|$.

(b) *If u and v are not contained in the same 3-residue of G and G' is obtained from G by fusion on u and v, then* $|G'| \simeq |G| \# (S^1 \otimes S^2)$.

(c) *If G' is obtained from G by removing a degenerate 2-dipole then* $|G| \simeq |G'| \# (S^1 \otimes S^2)$.

(d) *If G' is obtained from G by removing a degenerate 1-dipole then one of following cases holds*:

  (i) $|G| \simeq |G'|$

  (ii) $|G| \simeq |G'| \# (S^1 \otimes S^2)$.

  (iii) $|G| \simeq |G'| \# (S^1 \otimes S^2) \# (S^1 \otimes S^2)$

  (iv) $|G| \simeq |G_1'| \# |G_2'| \# (S^1 \otimes S^2)$

depending on whether the endpoints u and v of the dipole are both contained on exactly (i) *one 2-residue with colors other than that of edge* $\{u, v\}$; (ii) *two such 2-residues*; (iii) *three such 2-residues and G' is connected*; *or* (iv) *three such 2-residues and G' has two connected components G_1' and G_2'.*

Examples of parts (a) and (c) are shown in Figs 4 and 5 respectively. In Fig. 4, G_1 and G_2 both encode $S^1 \times S^2$ and $G_1 \overset{uv}{\#} G_2$ encodes the connected sum.

Some comments on Theorem 6 are in order. First, the graph G' always encodes a 3-manifold. In part (a) $|G_1 \overset{uv}{\#} G_2|$ is independent of u and v unless both G_1 and

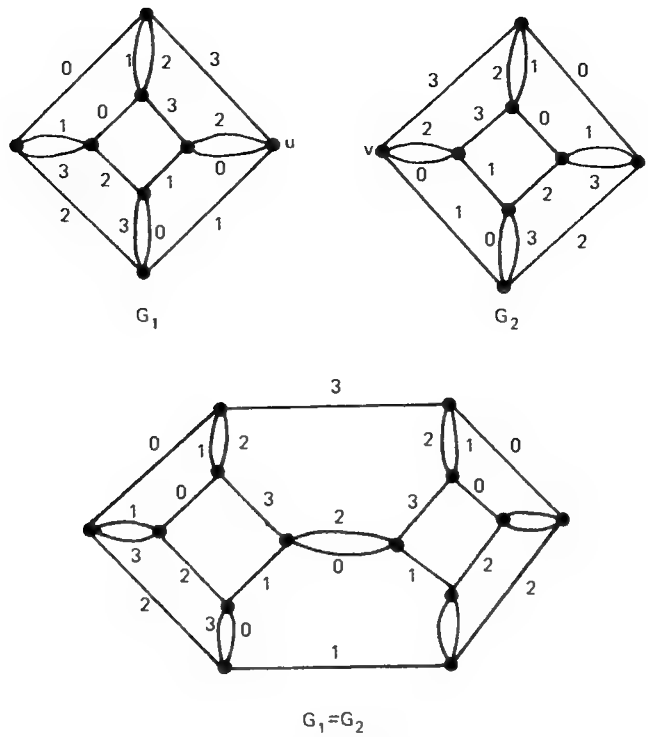

Fig. 4. Connected sum.

G_2 are bipartite. In this case both $|G_1|$ and $|G_2|$ are orientable (by Theorem 4) and there are two possible topological connected sums (orientable and non-orientable); which one depends on the partite sets in which u and v belong. In part (b) $\otimes$ is the orientable or non-orientable sphere bundle depending on whether or not there is a path from u to v with an odd number of edges. Likewise in (c) and (d) the graph G determines the type of bundle in a straightforward way, but we omit the details here and refer the reader to the original sources for these results. The only case of fusion not listed in Theorem 6 is that of a

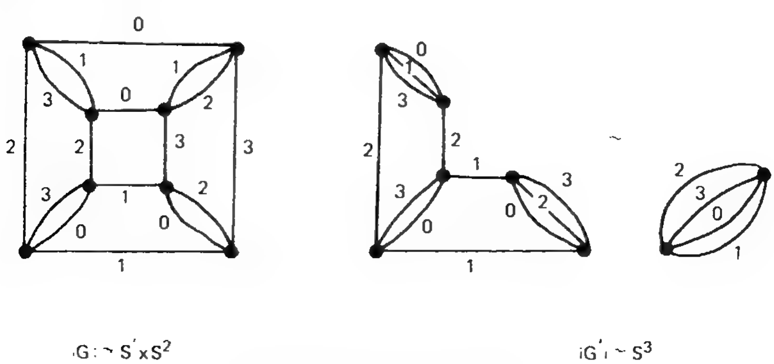

Fig. 5. Removing a handle.

degenerate 1-dipole where u and v are contained in *no* 2-residue (of type not containing the color of $\{u, v\}$). Unlike the cases covered in Theorem 6, there is no known characterization of the topological space corresponding to the 4-graph after the removal of such a dipole. Unfortunately, in a large class of 4-graphs, all dipoles are of this type [8].

Proof sketch of Theorem 6. In part (a) the operation $\overset{uv}{\#}$ corresponds, via Theorem 2, to removing a facet from ΔG_1 and ΔG_2 and identifying boundaries, i.e. a connected sum. Likewise in part (b) the fusion corresponds to removing two disjoint facets from ΔG and identifying boundaries. This is usually referred to as 'adding a handle', which is equivalent to a connected sum as in part (b). For part (c) a 4-graph G^* exists (see [12] for the construction) such that (1) G^* is equivalent to G' (by removing two nondegenerate dipoles of G^*) and (2) G is obtained from G^* by fusion of two points of G^* not in the same 3-residue. The result then follows from part (b) and Theorem 5. Part (d) is proved in a similar fashion using part (c). □

4. Canonical *n*-graphs

One of the major goals in the theory of n-graphs is to obtain canonical forms from PL-manifolds. Various notions of 'canonical' appear in the literature. If M is a PL-manifold, then G is called *minimum* for M if G is an n-graph with minimum number of points that encodes M [19]. For example, the minimum 4-graph for the 3-sphere is given in Fig. 6a (two 3-balls with their boundaries identified). The minimum number of $(n-1)$-residues in an n-graph is n. An n-graph that achieves this minimum is called *simple* [19]. A simple n-graph that encodes a manifold is also referred to in the literature as a *crystallization* [2]. We call an n-graph with no non-degenerate dipoles *reduced*. It is easy to check that if G is reduced, then G is simple. The converse is not true, as in Fig. 6b. Also it is obvious that if G is minimum, then G is reduced. Again the converse is not true, as in Fig. 6c [15]. By removing nondegenerate dipoles until no longer possible, Theorems 1 and 5 imply:

Corollary 7. *For any manifold M there is a reduced n-graph G such that $|G| \simeq M$.*

A strong form of Theorem 5 for simple n-graphs is proved in [4].

Theorem 8. *Let G_1 and G_2 be n-crystallizations. Then $|G_1| \simeq |G_2|$ if and only if G_2 can be obtained from G_1 by a finite sequence of the following moves:*

(a) *adding or removing a non-degenerate m-dipole, $n-1 > m > 1$.*

(b) *addition of a non-degenerate 1-dipole followed by the removal of another non-degenerate 1-dipole.*

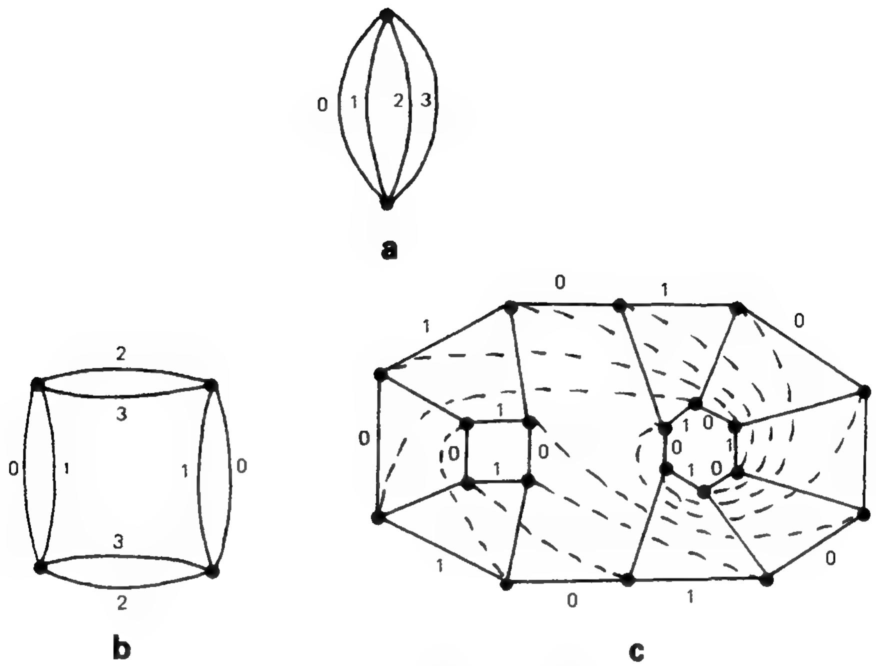

Fig. 6. Some 4-graphs encoding S^3.

Note that in each step in the process of Theorem 8, the intermediate n-graph is also a crystallization. Also for any $i \in [n]$, the moves (a) and (b) can be taken so that either all the dipoles contain color i or none contain color i [3].

The ideal situation would be to have a unique reduced graph to represent each equivalence class of n-graph. Unfortunately the only non-trivial case for which this is known is $n = 3$. To classify the 3-graphs up to equivalence let C_{2m} be the cycle with $2m$ points $\{0, 1, \ldots, 2m-1\}$ where edge $\{2i-1, 2i\}$ is colored 0 and $\{2i, 2i+1\}$ is colored 1 for all $i \pmod{2m}$. Let H_m be the 3-graph obtained from C_{2m}, m odd, $m \geq 1$, by adding edges $\{i, i+m\}$ colored 2 for all $i \pmod{2m}$. Likewise let L_m be obtained from C_{2m}, $m \geq 2$, by adding edges $\{i, 2m-i\}$ and $\{0, m\}$ colored 2 for all $i \pmod{2m}$. The graphs H_5 and L_5 are shown in Fig. 7.

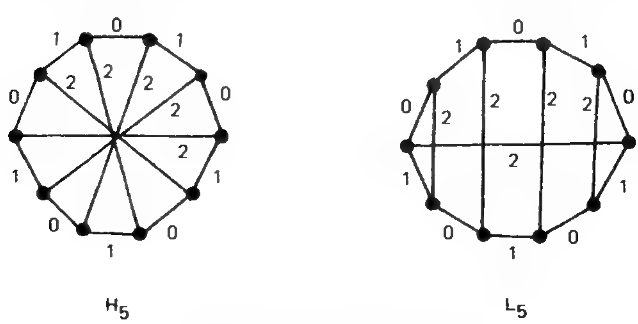

Fig. 7. Graphs encoding surfaces.

Theorem 9. *Each 3-graph is equivalent to a unique graph H_m, $m \geq 1$, m odd or L_m, $m \geq 2$.*

Corollary 10. *Two reduced 3-graphs are equivalent if and only if they have the same number of points and are both bipartite (or not bipartite).*

Theorem 9 is just a graph theoretic version of the well known classification of closed surfaces. Using Theorem 2 (to compute the Euler characteristic) and Theorem 4, it is easy to check that $|H_{3-\chi}|$ and $|L_{3-\chi}|$ are the orientable and nonorientable surfaces, resp., with Euler characteristic χ. So $|H_1|$ is homeomorphic to the 2-sphere, $|H_3|$ the torus, $|L_2|$ the projective plane, $|L_3|$ the Klein bottle, etc. Concerning Corollary 10, the number of points and the property of being bipartite are invariants of reduced 3-graphs because all steps in Theorem 8 must be of type (a). Each step of type (a) leaves the number of points of G and the properties of being reduced and bipartite invariant. In the other direction the corollary follows from Theorem 9. If G and G' are reduced, have an equal number of points and are both bipartite (or not bipartite), then this is also true of B and B', the graphs to which, by Theorem 9, they are equivalent. Then necessarily $B = B'$.

Any 3-manifold can be encoded by a 4-graph, so naturally the classification problem for 4-graphs becomes drastically more complex than for 3-graphs. For example, by Theorem 5 the 4-graphs in Fig. 6c and 6a must be equivalent, but there is no known algorithm, in general, for obtaining the correct sequence of dipole additions and removals. Certain n-graphs with additional structure, called normal crystallizations, have been shown to encode any 3-manifold [1, 14], but these also have not led to a classification. (So the somewhat technical definition is omitted here.) The determination of a set of moves to get, algorithmically, from any n-graph to any equivalent one would, of course, be a remarkable discovery. For more on this problem see [24].

5. The fundamental group

The encoding of a manifold M by an n-graph provides easy algorithms for finding a presentation of the fundamental group of M. We give two such algorithms. Here $\langle X \mid R \rangle$ denotes a presentation of a group with generators X and relations R. Let H be a subgraph of an n-graph G and e an edge in $E(G) - E(H)$. Call e *dependent* on H if there is a k-colored cycle, $k < n$, containing e, all of whose other edges lie in H. Let $H = H_0, H_1, \ldots, H_m = H^*$ be a sequence of subgraphs of G such that

(1) $H_{i+1} = H_i + e$ where e is dependent on H_i and

(2) there is no edge in G dependent on H^*.

Then H^* is called the *closure* of H in G.

In both algorithms the input is a manifold encoded by an n-graph and the output is a presentation for $\pi_1(|G|)$.

Algorithm 1 [24].

(0) Remove non-degenerate dipoles until G is reduced.
(1) Construct a spanning tree T in G.
(2) Determine the closure T^* of T in G.
(3) Arbitrarily assign an orientation to each edge in $X = E(G) - E(T^*)$.
(4) For each 2-residue τ let r_τ be the sequence of edges in X around the cycle τ, each with exponent $+1$ or -1 depending on the orientation. Let $R = \{r_\tau \mid \tau \text{ is a 2-residue}\}$.
(5) Then $\langle X \mid R \rangle$ is a presentation for $\pi_1(|G|)$.

Algorithm 2 [18].

(0) Remove non-degenerate dipoles until G is reduced.
(1) Let $X = V(G)$, the point set of G.
(2) Choose two colors $j, k \in [n]$ and let C denote the set of all 2-residues (cycles) of type $\{j, k\}$.
(3) For any cycle $\sigma \in C$ of length $2m$ let r_σ denote the word $x_1 x_2^{-1} \cdots x_{2m-1} x_{2m}^{-1}$ where $x_i \in X$ is the i-point of σ. The initial point x_1 and the direction around σ is arbitrary, except that the edge connecting x_1 and x_2 should be colored j. Let $R_1 = \{r_\sigma \mid \sigma \in C\}$.
(4) Let $R_2 = \{xy^{-1} \mid x, y \in X$ and x, y belong to the same residue of type $[n] - \{j, k\}\}$.
(5) If x_0 is an arbitrary point of G then $\langle X \mid R_1 \cup R_2 \cup \{x_0\} \rangle$ is a presentation of $\pi_1(|G|)$.

As an example, consider the 3-graph in Fig. 8, which encodes the Klein bottle K^2. (Those edges not colored 0 or 1 are understood to be colored 2.) Using Algorithm 1 (Fig. 8a), the spanning tree T consists of all edges except one in the unique residue of type $\{0, 1\}$ and T^* is the entire (darkened) cycle. Then the

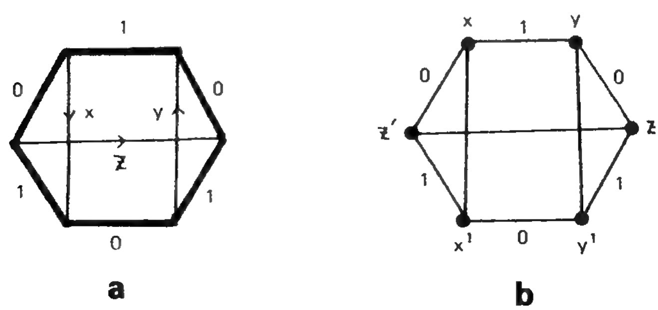

Fig. 8. Computing the fundamental group of $|G|$.

presentation of $\pi_1(K^2)$ is $\langle x, y, z \mid xyz^{-1} = xzy = 1\rangle$ which, by removing generator z, is equivalent to $\langle x, y \mid x^2y^2 = 1\rangle$.

Choosing the pair of colors {0, 1} for Algorithm 2 (Fig. 8b) the presentation is

$$\langle x, y, z, x', y', z' \mid x = x', y = y', z = z', xy^{-1}zy'^{-1}x'z'^{-1} = 1, z = 1\rangle$$

which simplifies after removing generators x', y', z', z to

$$\langle x, y \mid xy^{-1}y^{-1}x = 1\rangle = \langle x, y \mid x^2y^2 = 1\rangle.$$

Proof sketch of Algorithms 1 and 2. The first algorithm is based on the fact that G is the dual 1-skeleton of ΔG. Then up to homotopy, each based loop of $\pi_1(|G|)$ is represented by a based closed edge path in G. Each edge $e \in E(G) - E(T)$ represents the path in G that is the unique cycle in $T + e$. An edge in T^* represents a null homotopic loop in $|G|$; hence X is the generating set for $\pi_1(|G|)$. Each two colored cycle in G also corresponds to a null homotopic loop in $|G|$, hence the relations R.

The idea of Algorithm 2 is dual to that of Algorithm 1, in that loops in $|G|$ are represented by based edge paths in ΔG rather than in G. The fundamental group $\pi_1(|G|)$ is isomorphic to the standard edge path group $E(\Delta G)$ whose elements are closed edge paths in the 1-skeleton of ΔG, and edge paths around 2-simplices are null homotopic. It is the main theorem of [13] that a presentation $\langle X \mid R\rangle$ for $E(\Delta G)$ can be obtained by taking X as the set of all edges of type $\{j, k\}$ except one (recall that the vertices of ΔG are colored), and the relations are read around the links of simplices of type $[n] - \{j, k\}$. Translating this presentation, via Theorem 2, from ΔG to the graph G results in Algorithm 2. □

6. Regular embedding

Let F be a closed surface. An embedding of an n-graph G on F is called *regular* (strongly regular in [10]) if

(a) The components of $F - G$ are 2-cells.

(b) For any adjacent pair of points u and v of G, the cyclic permutation τ_u of the color set $[n]$ induced by the edges about u is the inverse of the cyclic permutation τ_v induced by the edges about v.

Up to inverse the cyclic permutation of the edge colors on F is the same at each point. So by counting the number of such cyclic permutations, it is easy to see that there is at most one regular embedding of a 3-graph and at most three regular embeddings of a 4-graph. Note that each 2-cell of F must be bounded by a 2-residue. Conversely, by spanning each of these (disjoint) 2-residues by a 2-cell and identifying pairs of edges that are the same in G, each of the regular embeddings above can be constructed. In general this argument shows that there are exactly $\frac{1}{2}(n-1)!$ regular embeddings of an n-graph.

In this section a graph theoretic invariant of a PL manifold is defined that

simultaneously generalizes the ordinary genus of a surface and the Heegaard genus of a 3-manifold. For an n-graph G let the *genus* $\rho(G)$ be the smallest integer g such that G regularly embeds in a surface of genus g. (For an orientable surface M the genus $g(M)$ is the number of torus factors in a connected sum (handles), while for a non-orientable surface it is the number of projective plane factors (cross caps). Then for a PL manifold M let

$$\rho(M) = \min\{\rho(G) \mid |G| \simeq M\}$$

be the minimum genus of any encoding of M. Using Theorem 4 it is not difficult to show that the surface F on which G is embedded is orientable if and only if $|G|$ is orientable. Hence $\rho(M)$ is the orientable or non-orientable genus depending on whether or not M is orientable.

The definition above and the results that follow appear in [11]. Recall that a handlebody of genus g is a 3-manifold (with boundary) obtained by identifying in pairs $2g$ disjoint 2-cells on the boundary of a 3-ball [16]. A *Heegaard splitting* of genus g of a 3-manifold M is a pair H, H' of handlebodies such that $H \cap H' = \partial H = \partial H'$ and $H \cup H' = M$. The common boundary $\partial H = \partial H'$ is called the *Heegaard surface*, which is a surface of genus g if orientable or $2g$ if not. The *Heegaard genus* $h(M)$ is the smallest integer g such that M admits a Heegaard splitting of genus g.

Theorem 11. (a) *If M is a 2-manifold then $\rho(M) = g(M)$.*
(b) *If M is a 3-manifold then*

$$\rho(M) = \begin{cases} h(M) & \text{if } M \text{ is orientable} \\ 2h(M) & \text{if } M \text{ is non-orientable.} \end{cases}$$

Proof sketch of Theorem 11. Part (a) of Theorem 11 is easy to prove because G, as the dual graph of the triangulation ΔG, is regularly embedded on $|G|$. Part (b) is proved in two parts. Given a 4-graph G regularly embedded on a surface F, assume, without loss of generality, that for each cyclic permutation τ_u we have $\tau_u^2(0) = 1$. Then consider the surface S that is the union of the 4-sided cells embedded in the facets of ΔG (see Fig. 9) such that there is a vertex of a 4-sided

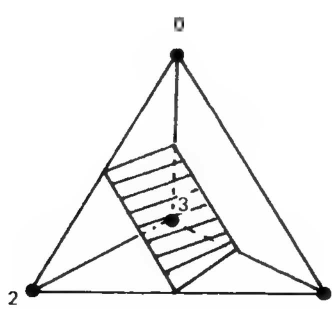

Fig. 9. Portion of the Heegaard surface.

cells at the midpoint of each edge of a facet except the edges 01 and 23. Now S splits $|G|$ into two handlebodies that are regular neighborhoods of the subgraphs of G induced by the edges colored 01 and 23, resp. Hence S is a Heegaard surface. The graph G is embedded in S as the dual graph of this cell division into squares such that the cyclic permutations about the vertices of this embedding are the same as for the embedding of G in F. Therefore S is homeomorphic to F.

Conversely the Heegaard splitting of genus g determines two sets of pairwise disjoint curves on the Heegaard surface F. Each such set consists of the boundaries of a set of pairwise disjoint 2-cells that cut the handlebody H or H' into a 3-ball. These systems of curves, called a Heegaard diagram, can be altered slightly to obtain the desired 4-graph embedding on the surface F. □

In [6] estimates are made on the genus of some 4-manifolds; here T_g, U_h denote the orientable surface of genus g and the non-orientable surface of genus h, respectively:

$$\rho(S^1 \times T_g) = 2g + 1$$

$$\rho(S^1 \times U_h) = 2h + 2$$

$$2(g + g') \leq \rho(T_g \times T_{g'}) \leq 2(6gg' + 3(g + g') + 2)$$

$$2(2g + h) \leq \rho(T_g \times U_h) \leq 2(6gh + 3(2g + h) + 4)$$

$$2(h + h') \leq \rho(U_h \times U_{h'}) \leq 2(3(hh' + h + h') + 4)$$

A corollary of Theorem 11 is that a 4-graph of genus 0 encodes a manifold $|G|$ that has Heegaard genus 0, and hence is a sphere. This is generalized in [5].

Theorem 12. *For an $(n + 1)$-graph G, $\rho(G) = 0$ if and only if $|G| \simeq S^n$.*

The proof of Theorem 12 is mainly topological. A stronger statement than this theorem is the following. If true, it should have a completely combinatorial proof. Here G_n^2 is the n-graph with exactly two points and n edges joining them. It is the minimum n-graph for S^{n-1}, (G_4^2 is shown in Fig. 6a).

Conjecture. *If G is a reduced n-graph and $\rho(G) = 0$ then $G = G_n^2$.*

7. Conclusion

The subject of n-graphs is new. Three-manifold theory is well developed, whereas the theory of 4-graphs is not. The hope is that the graph theoretic method holds some potential for combinatorial insight into topological questions. As the subject of n-graphs has developed the proofs have become less topological and more combinatorial. When it is not necessary to refer to the complex ΔG, via Theorem 2, the proofs (in the biased estimation of the author) take on a clear graph theoretic character.

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ON POINT-LINEAR ARBORICITY OF PLANAR GRAPHS

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The point-linear arboricity of a graph $G=(V, E)$, written as $\rho_0(G)$, is defined as $\rho_0(G)=\min\{k \mid$ there exists a partition of V into k subsets, $V=\bigcup_{i=1}^{k} V_i$, such that $\langle V_i\rangle$ is a linear forest for $1<i<k\}$. In this paper, we will discuss the point-linear arboricity of planar graphs and obtained following results:
$\rho_0(G)=2$ if G is a outplanar graph.
$\rho_0(G)=4$ if G is a planar graph.

1. Introduction

Chartrand [1] studied the point-arboricity of planar graphs and showed that the point-arboricity of a planar graph does not exceed 3.

Now we will discuss the point-linear arboricity. The point-linear arboricity of a graph $G=(V, E)$, written by $\rho_0(G)$, is defined as $\rho_0(G)=\min\{k \mid$ there exists a partition of V into k subsets, $V=\bigcup_{i=1}^{k} V_i$, such that $\langle V_i\rangle$ is a linear forest for $1<i<k\}$.

In the context, $\tilde{G}$ shows that $\tilde{G}\sim G$, i.e. G is homomorphic to G. Throughout this paper, all the graphs are simple, and we assume that $\tilde{K}_{2,3}$ is not isomorphic to K_4-e for $e\in E(K_4)$.

In this paper, it will be proved that $\rho_0(G)\leqslant 4$ if G is a planar graph.

2. Main results

Lemma 1. *G is a planar graph with order n.*

If $|\{v \mid v\in V(G)$ and $d(v)=n-1\}|>2$, then $\rho_0(G)<2$.

Proof. Take $v_1, v_2\in V(G)$ with $d(v_1)=d(v_2)=n-1$, then it is obvious that $d(v)\leqslant 4$ for $v\in V\setminus\{v_1, v_2\}$, $\langle V\setminus\{v_1, v_2\}\rangle$ does not contain cyclics for otherwise $\langle V(C)\cup\{v_1, v_2\}\rangle$ would be homomorphic to K_5. So, $\langle\{v_1, v_2\}\rangle$ and $\langle V\setminus\{v_1, v_2\}\rangle$ are linear forests. □

Lemma 2. *Let G be a maximal planar graph and $|V|>3$, then $d(v)\geqslant 3$ $\forall v\in V$.*

Proof. It is obvious, and therefore is omitted here.

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Lemma 3. *Let G be a maximal planar graph with $|V| = n > 6$, then* $\max_{v \in V} d(v) \geq 5$.

Proof. By Euler's formula we can obtain this conclusion.

It can be proved that if G is a planar graph with $|V| = n \leq 8$ then $\rho_0(G) < 2$. However, there exists a planar graph with $n = 9$, its point-linear arboricity is larger than 2. See the diagram in Fig. 1.

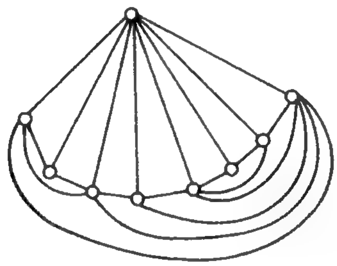

Fig. 1.

Theorem 2. *If G is an outplanar graph, then $\rho_0(G) \leq 2$.*

Broerer and Mynhardt [3] have given a proof of the theorem. Here we give another proof.

Proof. By Theorem 11, 12 in [2], G contains neither $\tilde{K}_4$ nor $\tilde{K}_{2,3}$. Without losing the generality, we assume G is connected and $|V| \geq 4$.

If $d(v) \leq 2$ for $\forall v \in V$, then the conclusion of Theorem 2 obviously holds.

We now assume that there exists a point $v_0 \in V$ such that $d(v_0) \geq 3$. Let $A_0 = \{v_0\}$, $A_1 = N(v_0)$ where $N(u) = \{v \mid (u, v) \in E\}$, $A_{i+1} = N(A_i) \setminus \{A_i \cup A_{i-1}\}$ for $i \geq 1$, where $N(A) = \bigcup_{v \in A} N(v)$.

As V is limited, there exists an integer t such that $A_t \neq \emptyset$ and $A_{t+1} = \emptyset$. Furthermore, we obtain a partition of V:

$$V = \bigcup_{i=0}^{t} A_i, \qquad A_i \cap A_j = \emptyset \quad \text{if } i \neq j.$$

We are prior to prove that $\forall i \in \{1, 2, \ldots, t\}$, $\langle A_i \rangle$ is a linear forest. Otherwise, there exists an A_i such that $\langle A_i \rangle$ contains either cyclics or $K_{1,3}$. Let us assume that $\langle A_t \rangle$ contains a cyclic C with $|V(C)| \geq 3$ and take $v_1, v_2, v_3 \in V(C)$. By the definition of partition we know that there exists a point $a \in \bigcup_{j=0}^{i-1} A_j$, such that $\forall \alpha \in \{1, 2, 3\}$ there exists a $(a - v_\alpha)$ path P_α in $\langle \bigcup_{j=0}^{i} A_j \rangle$ which satisfies the following properties:

$$V(P_\alpha) \cap V(P_\beta) = \{a\} \quad \text{if } \alpha \neq \beta.$$

$$V(P_\alpha) \cap A_t = \{v_\alpha\} \quad \text{if } \alpha = 1, 2, 3.$$

Thus, C, P_1, P_2 and P_3 form a subgraph which is homomorphic to K_4. A contradiction is obtained.

If $\langle A_i \rangle$ contains $K_{1,3}$, with the similar method, we can prove that G contains $\tilde{K}_{2,3}$. The same contradiction is obtained.

Let

$$A = \bigcup_{k=0}^{[t/2]} A_{2k}$$

$$B = \bigcup_{k=1}^{[t+1/2]} A_{2k-1}$$

By definition we know $E(A_i, A_j) = \emptyset$ if $|i-j| \geq 2$. So $\langle A \rangle$ and $\langle B \rangle$ are all linear forests. Hence $\rho_0(G) \leq 2$. □

In the following context, G is a simple planar graph, $V = \bigcup_{i=0}^{t} A_i$ denotes a partion of V which can be obtained, similar to the proof of Theorem 2, where t is determined by a given graph.

Lemma 4. *$\forall i \in \{1, 2, \ldots, t\}$, $\langle A_i \rangle$ is an outplanar.*

Proof. Otherwise, there is i such that $\langle A_i \rangle$ contains either $\tilde{K}_4$ or $\tilde{K}_{2,3}$. Let us assume that $\langle A_i \rangle$ contains a subgraph $H = \tilde{K}_4$ and denote the four points which are not subdivisible v_1, v_2, v_3, v_4. If $i = 1$ then $\langle V(H) \cup \{v_0\} \rangle$ homomorphic to K_5 is a contradiction. So, $i > 1$, by the definition of partion $V = \bigcup_{j=0}^{t} A_j$, one of following two cases will arise:

Case 1. There exists a point $a \in \bigcup_{k=0}^{i-1} A_k$ such that $\langle \bigcup_{j=0}^{i} A_j \rangle$ contains (a, v_α)-path P_α for $\alpha = 1, 2, 3, 4$, which satisfies the following properties:

$$V(P_\alpha) \cap V(H) = \{V_\alpha\} \text{ for } = 1, 2, 3, 4$$

$$V(P_\alpha) \cap V(P_\beta) = \{a\} \quad \text{if } \neq \beta.$$

Case 2. There exist two different points a and $b \in \bigcup_{k=0}^{i-1} A_k$ such that there exist two points, without losing the generality we assume v_1, v_2, such that $\langle \bigcup_{j=0}^{i} A_j \rangle$ contains (a, v_α)-path P_α for $= 1, 2$. Meanwhile, there exists (b, v_β)-path P_β for $\beta = 3, 4$ and (a, b)-path $P_{a,b}$. These paths satisfy the following properties:

$$V(P_1) \cap V(P_2) = \{a\}$$

$$V(P_3) \cap V(P_4) = \{b\}$$

$$V(P_\alpha) \cap V(P_\beta) = \emptyset \qquad \alpha = 1, 2, \qquad \beta = 3, 4.$$

$$V(P_{a,b}) \cap V(P_\beta) = \{a\} \quad \alpha = 1, 2$$

$$V(P_{a,b}) \cap V(P_\beta) = \{b\} \quad \beta = 3, 4$$

$$V(P_\alpha) \cap V(H) = \{V_\alpha\} \quad \alpha = 1, 2, 3, 4$$

$$V(P_{a,b}) \cap V(H) = \emptyset.$$

For Case 1, $H \cup P_1 \cup P_2 \cup P_3 \cup P_4$ is homomorphic to K_5. For Case 2, $H \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_{a,b}$ contains $\tilde{K}_{3,3}$. These two cases result in contradiction. So, $\langle A_i \rangle$ does not contain $\tilde{K}_4$.

Similarly, we can prove that $\langle A_i \rangle$ does not contain $\tilde{K}_{2,3}$. □

Theorem 3. *Let G be a planar graph, then* $\rho_0(G) \leqslant 4$.

Proof. We can obtain the conclusion immediately from Theorem 2 and Lemma 4. □

Corollary 1. Let G be a planar graph with $|V(G)| = n$. If there exists a point $v \in V$, such that $d(v) \geqslant n - 3$, then $\rho_0(G) \leqslant 3$.

Proof. It is obtained immediately from Lemma 4. □

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ON THE LITTLEWOOD–RICHARDSON RULE IN TERMS OF LATTICE PATH COMBINATORICS

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This work presents a simple proof of the Littlewood–Richardson rule on multiplying Schur functions using nonintersecting paths and a characterization of Schur functions by this rule.

1. Introduction

Since the work [11] of MacMahon was published, many papers on combinatorial structure in symmetric groups followed, among them, [1–4, 7, 9, 12, 14] and [17–21]. The algorithmic structure of the expansion of the product of several Schur functions was discussed in 1934 in the paper [9] of Littlewood and Richardson. The algorithm that appeared in this paper is called the Littlewood–Richardson rule. A good deal of effort has gone into validating this algorithm. The first attempt was by Robinson [12], but his proof was not satisfactory as is shown in MacDonald [10], p. 73 and Remmel–Whitney [14], p. 485. By applying the Robinson–Schensted–Knuth algorithm, White [21], Hillman–Grassel [6], Thomas [19] and Remmel–Whitney [14] recently gave some proofs of this rule.

The Littlewood–Richardson rule in [9] for multiplying Schur functions $S_\lambda(x)$ and $S_\mu(x)$ states that

$$S_\lambda(x)S_\mu(x) = \sum_{|v|=|\lambda|+|\mu|} c^v_{\lambda\mu} S_v(x), \tag{1}$$

where

$$c^v_{\lambda\mu} = \text{number of tableaux of shape } v/\lambda \text{ such that the word is a lattice permutation and the weight is } \mu. \tag{2}$$

Since the Schur function $S_\lambda(x)$ is determined only by the shape λ of the Young tableau, and each Young tableau uniquely corresponds to a line of stones by Sato [15, 16], in this paper the Schur function is treated as a line of stones. Then, the rules (1) and (2) of the multiplying Schur functions give a random walk of the stones as nonintersecting paths (as for this definition, see [5] and as for the recent works, see [4] and [8]). Thus, the Schur function is grasped as an operator to

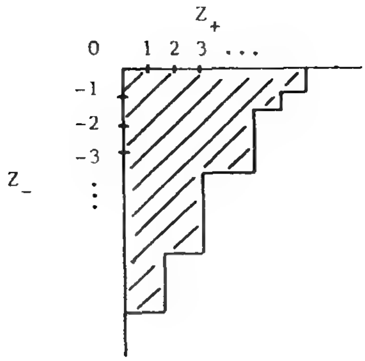


Fig. 1.

bring about a motion of the stones. As a result, the Jacobi-Trudy identity of the Schur function (cf. [10] p. 25) is shown.

2. Nonintersecting paths in the construction of Young tableaux

In this section, we discuss the correspondence between Young tableaux and arrangements of stones on the integer line (cf. [15] and [16]). In addition, we give a graph representation of the motion of these stones as nonintersecting paths.

We arrange all Young tableaux on the left corner in the product quarter space $Z_+ \times Z_-$ demonstrated in Fig. 1. Then the correspondence between Young tableaux and the arrangements of stones is given as follows (see Fig. 2);

(1) For the empty Young tableau, the corresponding line of stones is constructed by placing stones on all negative integers in the line,

(2) For the Young tableau with shape $(\lambda_1, 0, \ldots, 0, \ldots)$, the corresponding line of stones is constructed by moving up the stone on -1 of case (1) to the position $\lambda_1 - 1$,

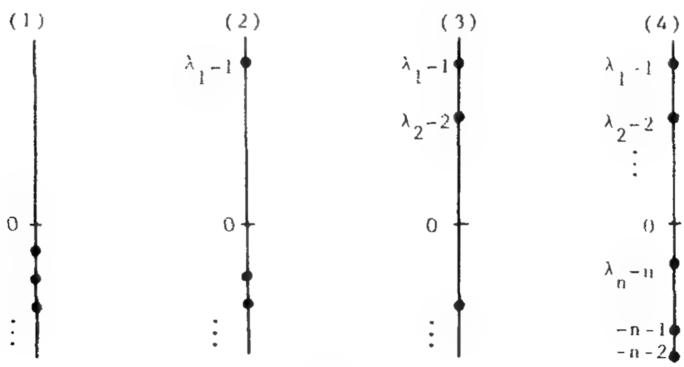


Fig. 2.

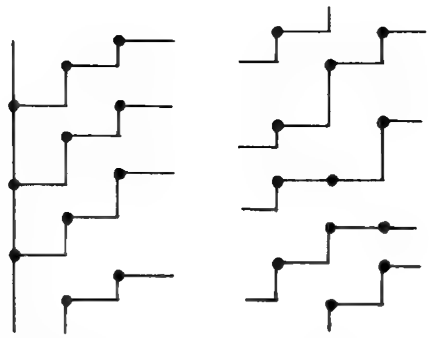


Fig. 3.

(3) For the Young tableau with shape $(\lambda_1, \lambda_2, 0, \ldots, 0, \ldots)$, the corresponding line of stones is constructed by moving up the stone on -2 of case (2) to the position $\lambda_2 - 2$.

Thus, proceeding in this manner, we obtain the lines of stones in the Young tableau with general shape $(\lambda_1, \lambda_2, \ldots, 0, \ldots, 0, \ldots)$;

(4) The stones on the line are placed on the positions denoted by integers $\{\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_n - n, -n-1, -n-2, \ldots\}$.

To represent the line of stones in the general case (4), we use the notation $\langle\lambda|$ for $\lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0, \ldots)$.

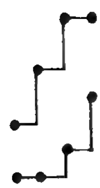
We now define the raising operator $[\mu_0]$, $\mu_0 \in Z$, on the space of the line of stones $\{\langle\lambda|;\ \lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots),\ \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0\}$.

Move each stone in the state $\langle\lambda|$ up from the lower part of the line of stones such that no stone displaces any stone above it and the total sum of the movements is equal to μ_0. The movements are denoted by the raising operator $[\mu_0]$ and the generated states are denoted by $\langle\lambda|\,[\mu_0]$. The application of the raising operators $[\mu_1], \ldots, [\mu_m]$, in order, leaves tracks which give a set of nonintersecting paths with m steps in the direction of the right (See Fig. 3). We denote the set of nonintersecting paths by $\langle\lambda|\,[\mu_1], \ldots, [\mu_m]$.

3. Nonintersecting path as words

In this section, we discuss the connection between words and nonintersecting paths, and we give a transformation rule for paths that preserves the property of nonintersection.

Let L be a nonintersecting path in the set $\langle\lambda|\,[\mu_1]\cdots[\mu_m]$. We denote the steps in L generated by the operator $[\mu_k]$ by the integer k. Let us arrange these integers in order from the right hand side of the topmost path in L. The sequence of these integers is called the word $\omega(L)$ of L. An example is given in Fig. 4.



$\omega(L) = 2211332.$

Fig. 4.

We assume the positive integers $\mu_1, \ldots, \mu_m$ associated with the raising operators $[\mu_1], \ldots, [\mu_m]$ are in the decreasing order, $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$.

Let us define the index $I_{ij}(L)$ of the word $\omega(L)$ with respect to i and j. Let $\omega(L_k)$ denote the subword in $\omega(L)$ generated by the first k steps. Then the index $I_{ij}(L_k)$ is defined as follows:

$$\begin{aligned} I_{ij}(L_k) = \{&\text{the number of occurrences of the letter } i \text{ in the word } \omega(L_k)\} \\ &- \{\text{the number of occurrences of the letter } j \text{ in the word } \omega(L_k)\}. \end{aligned}$$

Let the index $I_{ij}(L)$ with respect to i and j be the minimum of $I_{ij}(L_k)$ with respect to k:

$$I_{ij}(L) = \min_k . \, I_{ij}(L_k).$$

For $i<j$, the word $\omega(L)$ is called an (i, j)-lattice permutation if and only if the index $I_{ij}(L)$ is a non-negative integer. If the word $\omega(L)$ is an (i, j)-lattice permutation for any $i<j$, the path L is called a lattice permutation.

Let us state a rule given by Robinson [13], p. 46 for transforming non-$(i, i+1)$-lattice permutation paths in $\langle\lambda|\,[\mu_1]\ldots[\mu_m]$ onto $\langle\lambda|\,[\mu_1]\ldots[\mu_i+1][\mu_{i+1}-1]\ldots[\mu_m]$. We call it Robinson's recomposition rule with respect to $(i, i+1)$. It is composed of two parts as follows:

(1) If the index $I_{i,i+1}(L)$ for L in $\langle\lambda|\,[\mu_1]\ldots[\mu_m]$ is a negative integer, the letter $i+1$ in the first position k for which $I_{i,i+1}(L) = I_{i,i+1}(L_k)$ is changed into i.

(2) Let the path $\tilde{L}$ be in $\langle\lambda|\,[\mu_1]\ldots[\mu_i+1][\mu_{i+1}-1]\ldots[\mu_m]$. If the index $I_{i,i+1}(\tilde{L})$ is a nonpositive integer, the letter i next to the last position k for which $I_{i,i+1}(\tilde{L}) = I_{i,i+1}(\tilde{L}_k)$ is changed into $i+1$. Otherwise, the letter i in the lead position is changed into $i+1$.

Proposition 1. *Robinson's recomposition rules* (1) *and* (2) *preserve the property of nonintersection.*

Let the set of all nonintersecting paths which are lattice permutations in $\langle\lambda|\,[\mu_1]\ldots[\mu_m]$ be denoted by $\langle\lambda|\,[\mu_1, \ldots, \mu_m]$. We give a reduction formula which transforms nonlattice permutation paths into lattice permutation paths.

Proposition 2. *Let L be a nonintersecting path in $\langle\lambda|\,[\mu_1,\ldots,\mu_{m-1}][\mu_m]$ with negative index $I_{m-1,m}(L)=-k<0$.*

Then, Robinson's recomposition rule (1) *gives a unique path in $\langle\lambda|\,[\mu_1+\alpha_1,\ldots,\mu_{m-1}+\alpha_{m-1},\mu_m-k]$ for some $\alpha_i\in Z$, $i=1,\ldots,m-1$* and $|\alpha|=k$.

Conversely, for any path $\tilde{L}$ in $\langle\lambda|\,[\mu_1+\alpha_1,\ldots,\mu_{m+1}+\alpha_{m-1},\mu_m-k]$ with $\alpha_i\in Z_+$, $i=1,\ldots,m-1$ and $|\alpha|=k$, Robinson's recomposition rule (2) *gives a unique path L in $\langle\lambda|\,[\mu_1,\ldots,\mu_{m-1}][\mu_m]$ with negative index $I_{m-1,m}(L)=-k<0$.*

As a result, we have the following operator identity:

$$[\mu_1,\ldots,\mu_{m-1}][\mu_m]=\sum_{k=0}^{\mu_m}\sum_{k=|\alpha|}[\mu_1+\alpha_1,\ldots,\mu_{m-1}+\alpha_{m-1},\mu_m-k]. \tag{1}$$

4. Main result

Using the identity (1) repeatedly, we obtain the following result.

Theorem. *In the path correspondence of Proposition* 2, *we have the following operator identities*:

$$[\mu_1,\ldots,\mu_m]=\sum_{k=0}^{m-1}(-1)^k\sum_{i_1<\cdots<i_k\leqslant m-1}[\mu_1,\ldots,\mu_{i_1}+1,\ldots,\mu_{i_k}+1,\ldots,\mu_{m-1}][\mu_m-k], \tag{2}$$

$$[\mu_1,\ldots,\mu_m]=\sum_{\sigma\in S_n}\varepsilon(\sigma)[\mu_1+\sigma(1)-1]\ldots[\mu_m+\sigma(m)-m]. \tag{3}$$

Remark. Since there is a 1–1 correspondence between the Schur functions and the arrangement of stones, by applying the identity (3) to the state $\langle 0|$ of the empty Young tableau, we have the well known Jacobi–Trudy identity.

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CHROMATIC POLYNOMIALS OF GENERALIZED TREES

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This paper is a survey of results on chromatic polynomials of graphs which are generalizations of trees. In particular, chromatic polynomials of q-trees will be discussed. The smallest q-tree ($q \geq 1$) is the complete graph K_q on q vertices. A q-tree on $n+1$ vertices where $n \geq q$, is obtained by adding a new vertex adjacent to each of q arbitrarily selected, mutually adjacent vertices in a q-tree on n vertices. Another generalization of trees is the n-gon-trees. The smallest n-gon-tree ($n \geq 3$) is the n-gon which is a cycle on n vertices. A n-gon-tree with $k+1$ n-gons is obtained from a n-gon-tree with k n-gons by adding a new n-gon which has exactly one edge in common with any n-gon of a n-gon-tree with k n-gons.

1. Introduction

The graphs which we consider here are finite, undirected, simple and loopless. Let $P(G, \lambda)$ be the chromatic polynomial of the graph G. A graph G is uniquely m-colorable if and only if $p(G, m) = m!$. Two graphs are said to be chromatically equivalent if their chromatic polynomials are equal.

In [3], Chao and Whitehead proved that a large class of nonisomorphic connected graphs are chromatically equivalent. This class includes the cactus graphs; a *cactus* graph is a connected graph where each pair of cycles have no edge in common. In a planar graph, a cycle is a *mini-cycle* if and only if it is one of the two smaller cycles in every θ-subgraph. A *θ-subgraph* is a subgraph which consists of two cycles with exactly one common edge. A connected planar graph F is *forest-like* if each pair of cycles have at most one edge in common, and if the dual graph F^d of F is a forest where F^d is obtained from F by replacing each mini-cycle by a vertex, and by joining two vertices in F^d by an edge if and only if the corresponding mini-cycles in F have one edge in common. Chao and Whitehead proved that all forest-like connected planar graphs with n vertices, e edges and the same number of mini-cycles of each length are chromatically equivalent.

2. q-trees

The graphs called *2-trees* are defined by recursion. The smallest 2-tree is the complete graph K_2 on 2 vertices. A 2-tree on $n+1$ vertices (where $n \geq 2$) is obtained by adding a new vertex adjacent to each of 2 arbitrarily selected

adjacent vertices in a 2-tree on n vertices. (It should be noted that 2-trees are examples of forest-like connected planar graphs where the mini-cycles are triangles. A *triangle* is a cycle of length 3.) In [5], Skupień stated a problem to find a characterization of 2-trees. In [6], Whitehead proved that a graph G is a 2-tree on n vertices if and only if $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^{n-2}$. The proof technique is based on the fact that any n-vertex graph G having chromatic polynomial $\lambda(\lambda - 1)(\lambda - 2)^{n-2}$ is a uniquely 3-colorable graph. The method of proof is similar to the method used by Xu and Li, in [7], to prove that a wheel on an odd number of vertices is chromatically unique.

The graphs called *q-trees* are defined by recursion. The smallest q-tree is the complete graph K_q on q vertices. A q-tree on $n + 1$ vertices (where $n \geqslant q$) is obtained by adding a new vertex adjacent to each of q arbitrarily selected, mutually adjacent vertices of a q-tree on n vertices. In [2], Chao et al. proved that a n-vertex graph G is a q-tree if and only if $P(G, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - q + 1)(\lambda - q)^{n-q}$ where $n \geqslant q$. In their proof, they use the fact that any graph which is chromatically equivalent to a q-tree is a uniquely $(q + 1)$-colorable graph. They obtain a sharp upper bound estimate on the number of triangles in such a graph. They show that q-trees are the only graphs which obtain this upperbound. Every $(q + 1)$-coloring of a uniquely $(q + 1)$-colorable graph results in the same partition of its vertices into color classes. Chao et al. study the subgraphs induced by the union of any k of these color classes where $2 \leqslant k \leqslant q + 1$.

3. Trees of polygons

In [1], Chao and Li studied chromatic polynomials of “trees of polygons”. Let n be a positive integer where $n \geqslant 3$. The graphs called n-gon-trees are defined by recursion. The smallest n-gon-tree is the n-cycle, denoted C_n, which is the only two-connected graph containing n vertices and n edges. A n-gon-tree with $k + 1$ n-gons is obtained from a n-gon-tree with k n-gons by adding a new n-gon which has one edge in common with any n-gon of a n-gon-tree with k n-gons. Fig. 1 gives examples of a 3-gon-tree with six 3-gons, and a 4-gon-tree with five 4-gons. (In [6], Whitehead studied 3-gon-trees calling them 2-trees.) Chao and Li proved the following theorem which characterizes n-gon-trees.

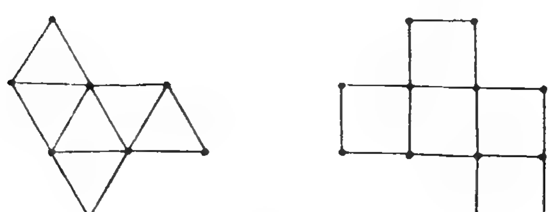

Fig. 1.

Theorem. *A graph G is a n-gon-tree with k n-gons (where integer $n \geqslant 3$ and integer $k \geqslant 1$) if and only if $P(G, \lambda) = \lambda(\lambda - 1)(Q(C_n, \lambda))^k$ where $Q(C_n, \lambda) = P(C_n, \lambda) \div (\lambda(\lambda - 1))$.*

Their proof of this theorem is based on Kuratowski's characterization of non-planar graphs in terms of K_5 and $K_{3,3}$ homeomorphs, Chao and Zhao's characterization (see [4]) of two-connected graphs containing no subgraphs homeomorphic to K_4, and a careful analysis of the coefficients of the chromatic polynomial of a n-gon-tree.

4. Unsolved problems

Problem 1. For each positive integer m, classify the uniquely m-colorable graphs. These graphs should yield new families of chromatically unique graphs.

Problem 2. Find a characterization of arbitrary trees of polygons similar to the characterization of n-gon-trees given by Chao and Li in [1]. By arbitrary trees of polygons, we mean trees of polygons where the length of the polygons are not required to be the same.

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PACKING OF GRAPHS—A SURVEY

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1. Introduction

Suppose $G_1, G_2, \ldots, G_k$ are graphs of order at most n. We say that there is a *packing* of $G_1, G_2, \ldots, G_k$ into the complete graph K_n if there exist injections $\alpha_i : V(G_i) \to V(K_n)$, $i = 1, 2, \ldots, k$ such that $\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \phi$ for $i \neq j$, where the map $\alpha_i^* : E(G_i) \to E(K_n)$ is induced by α_i. Similarly, suppose G is a graph of order m and H is a graph of order $n \geqslant m$ and there exists an injection $\alpha : V(G) \to V(H)$ such that $\alpha^*(E(G)) \cap E(H) = \phi$, then we say that there is a packing of G into H, and in case $n = m$, we also say that there is a packing of G and H, or G and H are packable. Thus G can be packed into H if and only if G is embeddable in the complement $\bar{H}$ of H. However, there is a slight difference between embedding and packing. In the study of embedding of a graph into another graph, usually at least one of the two graphs is fixed whereas in the study of packing of two graphs very often both the graphs are arbitrarily chosen from certain classes of graphs.

In this paper we shall concentrate mainly on some results on the following two packing problems. The first one is on dense packings of trees of different sizes into K_n. The second one is on general packings of two graphs having small size. Some open problems in this area will also be mentioned.

We shall use the following notation and definitions. The order, the size and the maximum valency of a graph G are denoted by $|G|$, $e(G)$ and $\Delta(G)$ respectively. A tree, a star and a path of order i are denoted respectively by T_i, S_i and P_i. The tree obtained from S_{n-1} $(n \geqslant 5)$ by inserting a new vertex on an edge is denoted by S_n'. The tree obtained from $S_{n-1}'(n \geqslant 6)$ by inserting a new vertex on the edge which is not incident with the vertex of maximum valency of S_{n-1}' is denoted by S_n''. We denote by $S(6)$ the tree obtained from S_4 by adding two new vertices each of which is joined to one end-vertex of S_4. The cycle of length m and the null graph of order r are denoted by C_m and 0_r respectively. The complete bipartite graph having bipartition V_1 and V_2 such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. The graph obtained from C_4 by adding an edge joining two opposite vertices is denoted by C_4^+. The disjoint union of two graphs G and H is denoted by $G \cup H$, and mG stands for the disjoint union of m copies of G. A graph of

order n and size m is called an (n, m) graph. If π is a packing of G into H, then the set of edges of H incident with at least one vertex of $\pi(x)$, $x \in V(G)$, is said to be *covered* by G.

Before we give our survey, let us point out that it has been shown that to find an efficient algorithm to pack two general graphs G and H is an NP-hard problem (see Garey and Johnson [15; p. 64]). However, if one restricts G and H to some special kind of graphs, for instance, both G and H are trees, then there exist polynomial time algorithms for the packing of G and H (mentioned in Hedetniemi et al. [17]).

2. Packing $n-1$ trees of different size into K_n

Erdös and Gallai [11] proved that every graph G having size $e(G) > \frac{1}{2}|G|(k-1)$ contains a path of length k. In 1963, Erdös and Sós made the following conjecture.

Conjecture (Erdös and Sós). *If G is a graph and $e(G) > \frac{1}{2}|G|(k-1)$, then G contains every tree T of size k.*

Motivated by the above conjecture, Gyárfás raised the question whether any sequence of trees $T_2, T_3, \ldots, T_n$ can be packed into K_n. Gyárfás' question has now been quoted in many books and research papers as the Tree Packing Conjecture (TPC).

Tree Packing Conjecture. *Any sequence of trees $T_2, T_3, \ldots, T_n$ can be packed into K_n.*

Till now, the TPC is still open. However, the TPC has been proved valid for quite a few cases. We list some of the results below.

Theorem 2.1 (Gyárfás and Lehel [16]). *Any sequence of trees $T_2, T_3, \ldots, T_n$, in which all but two are stars, can be packed into K_n.*

Theorem 2.2 (Gyárfás and Lehel [16]). *Any sequence of trees $T_2, T_3, \ldots, T_n$, where $T_i \in \{S_i, P_i\}$, can be packed into K_n.*

Theorem 2.3 (Hobbs and Kariraj; see Hobbs [18]). *Suppose each T_i, $2 \leq i \leq n$, with at most one exception, has diameter at most three. Then $T_2, T_3, \ldots, T_n$ can be packed into K_n.*

Theorem 2.4 (Straight [26]). *Suppose for each T_i, $2 \leq i \leq n$, with at most two exceptions, $\Delta(T_i) \geq i-2$. Then $T_2, T_3, \ldots, T_n$ can be packed into K_n.*

Theorem 2.5 (Straight [26]). *Suppose for each T_i, $2 \leq i \leq n$, with at most one exception, $\Delta(T_i) \geq i-3$. Then $T_2, T_3, \ldots, T_n$ can be packed into K_n.*

Theorem 2.6 (Straight [26]). *Suppose L_i is a tree of order i shown in* Fig. 1. *Then any sequence of trees $T_2, T_3, \ldots, T_n$, where $T_i \in \{S_i, P_i, L_i\}$, can be packed into K_n.*

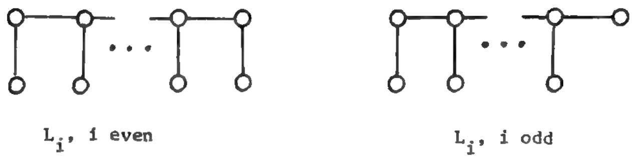

Fig. 1.

Theorem 2.7 (Fishburn [14]). *If each T_i is a double star, unimodal triple star, interior-*3 *caterpillar or thin-body spider, then $T_2, T_3, \ldots, T_n$ can be packed into K_n.*

(For definitions of the terms used in this theorem, see Yap [33].)

Straight [26] and Fishburn [14] verified that the TPC is true for all $n \leq 9$. The proof of Theorem 2.2 originally due to Gyárfás and Lehel was very complicated. A very nice, short proof of Theorem 2.2 has been found by Zaks and Liu [34].

Since a resolution to the TPC is hard, we may look for some other aspects of a similar nature. The following is a result of such an attempt.

Theorem 2.8 (Bollobás [3]). If $3 \leq s \leq \frac{1}{2}\sqrt{2}n$, *then any sequence of trees $T_2, T_3, \ldots, T_s$ can be packed into K_n.*

The following are some results concerning dense packings of paths into complete bipartite graphs.

Theorem 2.9 (Zaks and Liu [34]; Fink and Straight [13]). *Any sequence of paths $P_2, P_4, \ldots, P_{2n}$ can be packed into $K_{n,n}$.*

Theorem 2.10 (Zaks and Liu [34]; Fink and Straight [13]). *For odd n, the sequence of paths $P_3, P_5, \ldots, P_{2n+1}$ can be packed into $K_{n,n+1}$.*

We note that the above results on dense packings of trees having different sizes into K_n have been applied in the study of packing sequences of integers into a matrix. For details, see Hobbs [19].

Finally, we mention here another open conjecture on dense packing of trees.

Ringel's conjecture (Ringel [22]). *For any tree T_{n+1}, there is a packing of $2n+1$ copies of T_{n+1} into K_{2n+1}.*

Some progress on certain problems closely related to the Ringel's conjecture has been made. For details, we refer the readers to Huang and Rosa [20] and the references given therein.

3. Packing two graphs of small size

We next give a survey of the general packing of two graphs having small size.

Theorem 3.1 (Sauer and Spencer [23]). *Let G and H be graphs of order n. If $e(G)e(H) < \binom{n}{2}$, then G and H are packable.*

The example that $G = S_{2m}$ and $H = mK_2$ shows that Theorem 3.1 is best possible. However, if neither G nor H is a star, then perhaps, this theorem can be improved considerably.

The following theorem was first announced by Catlin [9]. A proof of this theorem was subsequently given by Sauer and Spencer [23] (see also Bollobás [2; P.425]).

Theorem 3.2. *Suppose G and H are graphs of order n. If $2\Delta(G)\Delta(H) < n$, then there is a packing of G and H.*

Bollobás and Eldridge [5] pointed out that the result of Theorem 3.2 is almost best possible. For example, suppose d_1 and d_2 are two integers such that $d_1 \leqslant d_2 < n$ and suppose $n \leqslant (d_1+1)(d_2+1) - 2$. For $i = 1, 2$, let $n = p_i(d_i+1) + r_i$, $1 \leqslant r_i \leqslant d_i + 1$. Let $G = p_1K_{d_1+1} \cup K_{r_1}$, and let $H = p_2K_{d_2+1} \cup K_{r_2}$. Then $\Delta(G) = d_1$, $\Delta(H) = d_2$ and there is no packing of G and H. Thus if $n \leqslant (\Delta(G)+1)(\Delta(H)+1) - 2$, then G and H may not be packable. Motivated by this example, they made the following conjecture.

Conjecture (Bollobás and Eldridge). *If G and H are graphs of order n and $(\Delta(G)+1)(\Delta(H)+1) \leqslant n+1$, then there is a packing of G and H.*

We remark that by Theorem 3.2, this conjecture is true when $\Delta(G) = 1$. It is mentioned in Bollobás [2; P. 426] that if this conjecture is true for $\Delta(G) = 2$ then it would imply a theorem of Corrádi and Hajnal ('On the maximal number of independent circuits in a graph,' Acta Math. Acad. Sci. Hungar. 14 (1963) 423–439) and that if this conjecture is true in general, then it would extend a theorem of Hajnal and Szemerédi (Proof of a conjecture of Erdös, in "Combinatorial Theory and its Applications", Vol. II (Eds., P. Erdös, A. Renyi and V. T. Sós), Colloq. Math. Soc. Bolyai 4, North-Holland (1970), 601–623). For details, see Catlin [10].

We next consider the problem of packing two graphs G and H in which $e(G) + e(H)$ is small. Milner and Welsh [21] noticed that if any two graphs G and

H of order n such that $e(G) + e(H) \leq [\frac{3}{2}(n-1)]$ are packable, then one can prove that the computational complexity of any graph property F has lower bound $[\frac{3}{2}(n-1)]$. They conjectured that $e(G) + e(H) \leq [\frac{3}{2}(n-1)]$ is sufficient for the packing of two graphs G and H. This conjecture was proved by Sauer and Spencer [23]. A generalization of this result is given below.

Theorem 3.3 (The main packing theorem of Bollobás and Eldridge [5]). *Suppose H and G are graphs of order n, $\Delta(H)$, $\Delta(G) < n-1$, $e(H) + e(G) \leq 2n-3$ and $\{H, G\}$ is not one of the following pairs: $\{2K_2, 0_1 \cup K_3\}$, $\{0_2 \cup K_3, K_2 \cup K_3\}$, $\{3K_2, 0_2 \cup K_4\}$, $\{0_3 \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, 0_3 \cup K_4\}$, $\{0_4 \cup K_4, K_2 \cup 2K_3\}$ and $\{0_5 \cup K_4, 3K_3\}$. Then there is a packing of G and H.*

The proof of the main packing theorem of Bollobás and Eldridge depends heavily on Lemma 3.4 given below. An alternative proof of Lemma 3.4 and a slight simplification of the original proof of Theorem 3.3 can be found in Teo [28] (see also Yap [33]).

Lemma 3.4 (Bollobás and Eldridge [5]). *Let T be a tree of order p and let G be a graph of order n. Suppose $2 \leq 2p \leq n$ $(n \geq 5)$, $\Delta(G) < n-1$ and $n-1 \leq e(G) \leq n + (n/p) - 3$. Then there is a packing of T and G such that T covers at least $p+1$ edges of G and $\Delta(G-T) < n-p-1$.*

The main packing theorem of Bollobás and Eldridge extends an earlier result of Burns and Schuster [7] (who proved that every $(n, n-2)$ graph can be embedded in itself). This theorem, in turn, has been extended to the case that $e(H) + e(G) \leq 2n-2$ where $n = |H| = |G|$. The extension is carried out in several steps. A preliminary step is to show that any tree T of order $n \geq 5$ can be packed into an $(n, n-1)$ graph G. This result (Theorem 3.5) is due to Slater et al. [25]. It generalizes an earlier result of Hedetniemi et al. [17]. The original proof of Theorem 3.5 uses induction on n by deleting two vertices from both T and G, and as a result many cases (when $n = 5$, 6) have to be verified. An alternative proof of Theorem 3.5, using induction on n by deleting only one vertex from both T and G, was later given by Teo [28] (see also Yap [33]).

Theorem 3.5 (Slater, Teo and Yap [25]). *Let T be a tree of order $n \geq 5$ and let G be an $(n, n-1)$ graph. If neither T nor G is a star, then there is a packing of T and G.*

An extension of Theorem 3.5 were independently obtained by Schuster [24] as well as Teo and Yap [29]. The generalization is as follows.

Theorem 3.6 (Schuster [24]; Teo and Yap [29]). *Suppose T is a tree of order $n \geq 5$ and G is an (n, n) graph such that $\Delta(T)$, $\Delta(G) < n-1$. If $\{T, G\} \neq \{P_5, 0_1 \cup$*

$C_4^+\}$, $\{P_6, 0_2 \cup K_4\}$, $\{S(6), 0_2 \cup K_4\}$, *or if* $G = \cup C_i$ *where* $i \geq 4$ *for at least one* i *and* $T \neq S_n'$ *or if* $G = kC_3$ *and* $T \neq S_n'$ *or* S_n'', *then there is a packing of* T *and* G.

Our next step to extend the main packing theorem of Bollobás and Eldridge is to find the forbidden pairs for the packing of two $(n, n-1)$ graphs G and H (Theorem 3.7). For $n \geq 5$, there are thirteen forbidden pairs. We shall adopt the notation shown in Fig. 2.

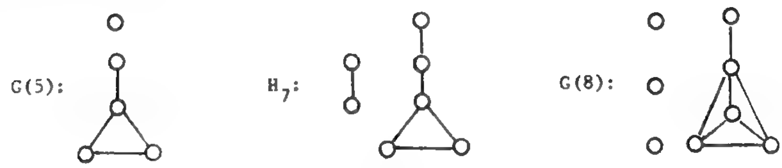

Fig. 2.

Theorem 3.7 (Teo and Yap [29]). *Suppose* G *and* H *are two* $(n, n-1)$ *graphs,* $n \geq 5$, *which are not stars. If* $\{G, H\}$ *is not one of the following thirteen pairs*:
(1) $\{P_2 \cup K_3, P_2 \cup K_3\}$, (2) $\{0_1 \cup C_4, 0_1 \cup C_4\}$, (3) $\{G(5), P_2 \cup C_3\}$, (4) $\{0_3 \cup K_4, P_2 \cup C_5\}$, (5) $\{0_3 \cup K_4, P_4 \cup K_3\}$, (6) $\{0_3 \cup K_4, H_7\}$, (7) $\{0_1 \cup 2K_3, S_4 \cup K_3\}$, (8) $\{0_1 \cup 2K_3, 0_3 \cup K_4\}$, (9) $\{0_1 \cup 2K_3, 0_1 \cup 2K_3\}$, (10) $\{G(8), P_2 \cup 2K_3\}$, (11) $\{0_2 \cup P_2 \cup K_4, P_2 \cup 2K_3\}$, (12) $\{0_6 \cup K_5, P_2 \cup 3K_3\}$, *and* (13) $\{K_3 \cup S_{n-3}, K_3 \cup S_{n-3}\}$, $n \geq 8$, *then there is a packing of* G *and* H.

We note that Theorem 3.7 generalizes an earlier result of Burns and Schuster [8] (who gave a complete characterization of packing an $(n, n-1)$ graph, $n \geq 4$, with itself) and Theorem 3.5.

The forbidden pairs $\{G, H\}$ for packing two graphs G and H of order n for which $e(G) + e(H) \leq 2n - 2$ have been completely determined. Besides the forbidden pairs given in Theorems 3.3 and 3.7, there are thirty more such pairs. Because there are so many forbidden pairs, for simplicity, we shall state the result only for $n \geq 9$. We require the notation shown in Fig. 3.

Fig. 3.

Theorem 3.8 (Teo and Yap [30]). *Suppose* G *and* H *are graphs of order* $n \geq 9$ *such that* $\Delta(G)$, $\Delta(H) < n - 1$ *and* $e(G) + e(H) \leq 2n - 2$. *If* $\{G, H\}$ *is not one of the following thirteen pairs*:
$\{3K_3, 0_5 \cup K_4\}$, $\{3K_2 \cup K_3, 0_4 \cup K_5\}$, $\{0_5 \cup K_4, K_2 \cup K_3 \cup K_4\}$, $\{0_5 \cup K_4, H_9\}$, $\{G(9), 3K_3\}$, $\{0_3 \cup K_2 \cup K_4, 3K_3\}$, $\{0_6 \cup K_4, 2K_3 \cup K_4\}$, $\{2K_2 \cup 2K_3, 0_5 \cup K_5\}$,

$\{0_6 \cup K_5,\ K_2 \cup 3K_3\}$, $\{0_7 \cup K_5,\ 4K_3\}$, $\{0_1 \cup S_{n-1},\ \cup C_i\}$, $\{K_2 \cup S_{n-2}, kC_3\}$ and $\{K_3 \cup S_{n-3},\ K_3 \cup S_{n-3}\}$, then G and H are packable.

Finally, we mention two results on packing two graphs G and H of sufficiently large order. Let $t_{s-1}(n)$ be the size of the Turan graph $T_s(n)$. Then $t_{s-1}(n)$ is approximately equal to $\frac{1}{2}[1 - 1/(s-1)]n^2$.

Theorem 3.9 (Bollobás and Eldridge [5]; see also Bollobás [2; P.426]). *Let m and s be positive integers such that*

$$\binom{s}{2} \leq m < \binom{s+1}{2}.$$

Let G and H be graphs of order n, $e(G) = m$ and

$$e(H) \leq \binom{n}{2} - t_{s-1}(n) - 1$$

If n is sufficiently large, then there is a packing of G and H.

Note that the example $G = K_s \cup 0_{n-s}$ and $H = \overline{T_{s-1}(n)}$ shows that Theorem 3.9 is best possible.

Theorem 3.10 (Bollobás and Eldridge [5]; see also Bollobás [2; P.427]). *Suppose $0 < \alpha < \frac{1}{2}$. If G and H are graphs of order n, $e(G) \leq \alpha n$, $e(H) \leq \frac{1}{5}(1-2\alpha)n^{3/2}$, and n is sufficiently large, then there is a packing of G and H.*

Note that the example $G = K_{t+1} \cup 0_{n-t-1}$ and $H = tK_{n/t}$ shows that Theorem 3.10 is near to being best possible.

Further results, conjectures and open problems on the packing of two graphs can be found in Bollobás [2; 436–437], and Bollobás and Eldridge [4, 5, 6].

4. Miscellanea

In this section we mention a few results on some other packing problems.

4.1 *Packing a graph with itself*

Burns and Schuster [7] proved that every $(n, n-2)$ graph can be packed with itself. They also found all the forbidden $(n, n-1)$ graphs G which cannot be packed with itself [8]. We have noted earlier that these results follow immediately from Theorems 3.3 and 3.7. In [12], Faudree et al. gave a complete characterization of packing (n, n) graphs $(n \geq 5)$ with itself. They also proved that if G is a graph of order n such that $e(G) \leq \frac{6}{5}n - 2$, $G \neq S_n$ and G contains no cycles of

length 3 or 4, then G can be packed with itself. Motivated by this result, they made the following conjecture.

Conjecture (Faudree, Rousseau, Schelp and Schuster [12]). *Each graph which is not a star and contains no cycles of length* 3 *or* 4 *as subgraphs is embeddable in its complement.*

4.2 *Packing digraphs*

Benhocine et al. [1] as well as Wojda and Wozniak [32] studied the packings of digraphs (directed graphs). Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ of the same order are packable if there exists a bijection $\alpha: V_1 \to V_2$ such that if $(x, y) \notin A_1$, then $(\alpha(x), \alpha(y)) \in A_2$. In [1], it has been proved that if $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are digraphs of order n such that $|A_1|\,|A_2| < n(n-1)$, then D_1 and D_2 are packable. From this, it follows that if $|A_1| + |A_2| \leq 2n - 2$, then D_1 and D_2 are packable. A complete characterization for packing two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ of order n for which $|A_1| + |A_2| \leq 2n - 1$ is also obtained in [1]. In [31], Wojda made the following conjecture.

Conjecture (Wojda [31]). *Let k and n be integers satisfying $1 \leq k \leq n(n-1)$. Denote by $f(n, k)$ the minimal number such that there exist digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ of order n, with $|A_1| = k$ and $|A_2| = f(n, k)$, for which there is no packing of D_1 and D_2. Then for every m satisfying $2 \leq m \leq n/2$, $f(n, n-m) = 2n - [n/m]$. In particular, $f(n, n-2) = 2n - [n/2]$.*

5. Some remarks on the Erdös-Sós conjecture

In Section 2 we mentioned that a result of Erdös and Gallai [11] confirms that the Erdös-Sós conjecture is true for $T = P_{k+1}$. The assumption that $e(G) > \frac{1}{2}|G|(k-1)$ implies that G has at least one vertex of valency k. Hence the Erdös-Sós conjecture is also true for $T = S_{k+1}$. It is not difficult to prove that the Erdös-Sós conjecture is also true for $T = S'_{k+1}$. It would be interesting to prove that the Erdös-Sós conjecture is true for some other types of trees.

In a recent paper, Zhou [35] proved that the Erdös-Sós conjecture is true for $k = |G| - 1$. In fact, for the cases $k = |G| - 1$ and $k = |G| - 2$, the Erdös-Sós conjecture follows immediately from Theorem 3.5. To show this, we first note that if $e(g) > \frac{1}{2}v(k-1)$ where $v = |G|$, then $e(\bar{G}) < \frac{1}{2}v(v-k)$. Thus Erdös-Sós conjecture can be restated as follows.

Erdös-Sós conjecture. *Suppose G is a graph of order v and T is any tree of size k.* If *$e(G) < \frac{1}{2}v(v-k)$, then T and G are packable.*

Now if $k = v - 1$ or $k = v - 2$, then $e(G) \leq v - 1$. Hence, by Theorem 3.5 (we assume that T is not a star), if $v \geq 5$, then T and G are packable. Thus Erdös-Sós

conjecture is true for $k = v - 1$ and $k = v - 2$. (The case that $v \leq 4$ can be verified easily.)

The above proof technique for the case $k \geq v - 2$ fails for other values of k. It would be interesting to develop a new proof technique to settle the case $k = v - 3$.

In his lecture at this conference, Szemerédi [27] mentioned that M. Ajtai, J. Komlós and himself have used the Uniformity Lemma to prove that if G is a graph and $e(G) > \frac{1}{2}|G|(k - 1)$, then G contains any tree of order $(1 - \varepsilon)k$ for small ε and large k.

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Z-TRANSFORMATION GRAPHS OF PERFECT MATCHINGS OF HEXAGONAL SYSTEMS

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Let H be a hexagonal system. We define the Z-transformation graph $Z(H)$ to be the graph where the vertices are the perfect matchings of H and where two perfect matchings are joined by an edge provided their symmetric difference is a hexagon of H. We prove that $Z(H)$ is a connected bipartite graph if H has at least one perfect matching. Furthermore, $Z(H)$ is either an elementary chain or graph with girth 4; and $Z(H) - V_m$ is 2-connected, where V_m is the set of monovalency vertices in $Z(H)$. Finally, we give those hexagonal systems whose Z-transformation graphs are not 2-connected.

In the past few years several kinds of transformation graphs were introduced such as tree graph [1], minimum tree graph [2], perfect matching polyhedra [3], matroid basis graph [4], Euler tour graph [5], and so on. In such a graph a vertex is a special sort of subgraph of a specified graph, and two vertices are joined by an edge provided that they can transform to each other by some specified transformation. Almost all of the above mentioned transformation graphs can be shown as polyhedra of $(0, 1)$. Their properties have been extensively studied [6].

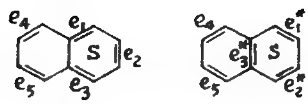
Recently some transformation graphs were studied and proved not to be polyhedra of $(0, 1)$, such as generalized directed tree graphs [7] and hierarchical tree of Kekulé patterns of hexagonal systems [8].

In this paper we introduce a new kind of transformation graph, namely Z-transformation graph of perfect matchings of hexagonal systems and observe some of its properties. Note that this kind of transformation is not polyhedra of $(0, 1)$.

A hexagonal system, also called "honeycomb system" or "hexanimal" (see e.g. [9]) is a finite connected plane graph with no cut-vertices in which every interior region is surrounded by a regular hexagon of side length 1 [10]. A perfect matching of a graph G is a set of disjoint edges of G covering all vertices of G.

In the present paper we confine oursleves to those hexagonal systems which have at least one perfect matching.

Definition 1. Let H be a hexagonal system with perfect matchings. The



$$M_1=\{e_1,e_2,e_3,e_4,e_5\},$$

$$M_2=\{e_1^*,e_2^*,e_3^*,e_4,e_5\},$$

$$M_1 \Delta M_2 = s\ .$$

Fig. 1.

Z-transformation graph $Z(H)$ is the graph where the vertices are the perfect matchings of H and where two perfect matchings M_1 and M_2 are joined by an edge provided that their symmetric difference $M_1 \triangle M_2$ is a hexagon of H. Let s be a hexagon of H and $M_1 \triangle M_2 = s$. Then $M_1 \triangle s = M_2$ and $M_2 \triangle s = M_1$. We call that M_1 and M_2 can be obtained from each other by a Z-transformation (see Fig. 1).

Let $s_1, s_2, \ldots, s_t$ be distinct hexagons of H. We use $H[s_1, s_2, \ldots, s_t]$ to denote the graph induced by the edges of $s_1, s_2, \ldots, s_t$.

Theorem 2. *Let H be a hexagonal system with perfect matchings. Then $Z(H)$ is bipartite.*

Proof. If $Z(H)$ does not contain a cycle, then $Z(H)$ is bipartite.

Now assume that $Z(H)$ contains a cycle $C = M_0 M_1 \ldots M_t$, where $M_0 = M_t$. Thus there exists a series of hexagons $s_1, \ldots, s_t$ such that $M_p = M_{p-1} \triangle s_p$ for $p = 1, \ldots, t$. Suppose that $s^1, \ldots, s^{m-1}$ and s^m are distinct hexagons among them, and s^i $(i = 1, \ldots, m)$ appears $\delta(s^i)$ times in $\{s_1, \ldots, s_t\}$. We now show that $\delta(s^i)$ is an even number for $i = 1, \ldots, m$. Note that if s^i contains an edge lying on the perimeter of H, then $\delta(s^i)$ must be even since $M_t = (\ldots((M_0 \triangle s_1) \triangle s_2) \ldots) \triangle s_t) = M_0$. Now suppose each edge of s^i does not lie on the perimeter of H. Since C is a cycle, there is no loss in generality in assuming that $s^i = s_1$. Let edge $e \in s_1$, and the other hexagon containing the edge e be s^*. Then we have $\delta(s^i) + \delta(s^*)$ is even. Assume that $\delta(s^i)$ is odd. Thereby $\delta(s^*)$ is odd too. Let $e^* \neq e$, $e^* \in s^*$, and e^* is parallel to e. Denote the other hexagon containing e^* by s^{**}. Then $\delta(s^{**})$ is odd too. Repeat this discussion, since H is finite, we eventually reach a hexagon $s' \in \{s^1, \ldots, s^m\}$ such that s' contains an edge lying on the perimeter of H and $\delta(s')$ is odd. This is contrary to the fact that for each hexagon s in $\{s^1, \ldots, s^m\}$ containing an edge lying on the perimeter of H, $\delta(s)$ must be even. This contradiction shows that $\delta(s^i)$ is even. Therefore $t = \sum_{i=1}^{m} \delta(s^i)$ is even, i.e. C is an even cycle.

The proof is completed. □

Theorem 3. *Let H be a hexagonal system with perfect matchings. Then $Z(H)$ is connected.*

Proof. From the theorem in [8], we can deduce that any perfect matching of H is connected with the special perfect matching of H (called root Kekulé pattern), since every simultaneous rotation can be considered as a series of rotations of a single hexagon. So $Z(H)$ is connected.

In order to simplify the discussion, a hexagonal system is to be placed on a plane so that a pair of edges of each hexagon lie in parallel with the vertical line. Before continuing we review briefly some results about hexagonal systems. Let M be a perfect matching of a hexagonal system H. If six vertices of a hexagon s of H are covered by three edges of M, i.e. s is an M-alternating cycle, then s is called respectively, proper and improper sextet [12] (see Fig. 2). For each hexagonal system with perfect matchings, there exist exactly two perfect matchings, one of which has only proper sextets, the other of which has only improper sextets [12]. From this we can deduce that for each hexagonal system H, the Z-transformation graph $Z(H)$ has at most two vertices of monovalency. □

Fig. 2.

Let V_m be the set of monovalency vertices of $Z(H)$. Then we have the following.

Theorem 4. *Let H be a hexagonal system with perfect matchings. Then $Z(H)$ is either a path or a graph of girth 4 and $Z(H) - V_m$ is 2-connected.*

Proof. Suppose $Z(H)$ is not a path. Let $M \in V(Z(H)) - V_m$. We first show that H has at least two disjoint hexagons which are M-alternating cycles. If M has valency greater than two in $Z(H)$, then the above conclusion is evident. Now suppose that M has valency two in $Z(H)$, i.e. there are only two M-alternating cycles which are hexagons of H, say s_1 and s_2. If s_1 and s_2 are edge disjoint, then there is nothing to prove. The remainder case is that s_1 and s_2 have an edge in common (see Fig. 3). Let $M_1 = M \triangle s_2$. It is not difficult to verify that for M_1,

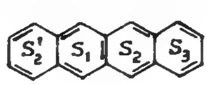


Fig. 3.

besides s_2, there is at most one hexagon, say s_3, which is an M_1-alternating cycle. In other words, M_1 has valency at most two in $Z(H)$. The same is true of $M'_1 = M \triangle s_1$, i.e. there is at most one hexagon besides s_1, say s'_2, which is an M'_1-alternating cycle. Note that $s_1 \cap s_2$, $s_2 \cap s_3$ and $s_1 \cap s'_2$ are in parallel with one another (see Fig. 3). Repeat this discussion, since H is finite, we can find a path $M'_r M'_{r-1} \ldots M'_1 M M_1 \ldots M_{t-1} M_t$ ($r \geq 1, t \geq 1$) which is a component of $Z(H)$. By the connectivity of $Z(H)$, $Z(H)$ is a path itself and this contradicts the hypothesis. Therefore, M has at least two disjoint hexagons which are M-alternating cycles. This implies that if $Z(H)$ is not a path, for any $M \in Z(H) - V_m$ M is contained in a 4-cycle of $Z(H)$, namely $MM_1M_2M_3$, where $m_1 = M \triangle s_1$, $M_2 = M_1 \triangle s_2$ and $M_3 = M_2 \triangle s_1$, and s_1 and s_2 are edge disjoint hexagons of H which are M-alternating cycles. Since $Z(H)$ is bipartite, the girth of $Z(H)$ is 4.

In the following we shall prove that $Z(H) - V_m$ is 2-connected. Obviously, $Z(H) - V_m$ is connected. It suffices to prove the conclusion: for any 2-path of $Z(H)$, say $M_1M_2M_3$, there is another path $M_1M'_2 \ldots M_3$ joining M_1 and M_3 which is internally vertex disjoint with $M_1M_2M_3$. In fact, if the above conclusion holds but $Z(H) - V_m$ is not 2-connected, then there is a cut vertex M of $Z(H) - V_m$. Take two vertices M' and M'' from different components of $Z(H) - V_m - M$ which are adjacent to M in $Z(H)$. Then there is no other path connecting M' and M'', a contradiction.

Now let $M_1M_2M_3$ be a 2-path in $Z(H)$, and $M_2 = M_1 \triangle s_1$, $M_3 = M_2 \triangle s_2$, where s_1 is a hexagon of H which is an M_1-alternating cycle, and s_2 is a hexagon of H which is an M_2-alternating cycle. If s_1 and s_2 are edge disjoint, then $M_1M'_2M_3$ is another path joining M_1 and M_3, where $M'_2 = M_1 \triangle s_2$ and $M_3 = M'_2 \triangle s_1$. If s_1 and s_2 are edge joint, then by previous discussion, there is another hexagon s_3 which is M_1-alternating cycle and is edge disjoint with s_1. If s_3 and s_2 are also edge disjoint, then there is another path $P' = M_1M'_2M'_3M'_4M_3$ joining M_1 and M_3, where $M'_2 = M_1 \triangle s_3$, $M'_3 = M'_2 \triangle s_1$, $M'_4 = M'_3 \triangle s_2$ and $M_3 = M'_4 \triangle s_3$. If s_3 and s_2 have an edge in common, since the degree of M_3 in $Z(H)$ is greater than 1, there is another hexagon s_4 which is an M_3-alternating cycle. It is not difficult to check that s_4 and s_2 are edge disjoint. If s_4 and s_1 are also edge disjoint, then we obtain a path as before joining M_1 and M_3. If s_4 and s_1 have an edge n in common (see Fig. 4), we can get another path $P'' = MM'_1M'_2M'_3M'_4M'_5M_3$, where $M'_1 = M \triangle s_3$, $M'_2 = M'_1 \triangle s_1$, $M'_3 = M'_2 \triangle s_4$, $M'_4 = M'_3 \triangle s_3$, $M'_5 = M'_4 \triangle s_2$ and $M_3 = M'_5 \triangle s_4$. The proof is complete. □

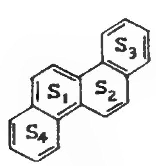


Fig. 4.

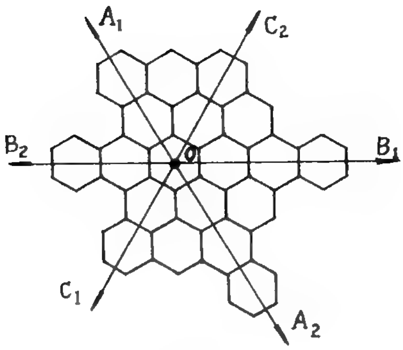


Fig. 5.

By the above theorem, a Z-transformation graph with $V_m = \emptyset$ is 2-connected. We now turn our attention to investigate when V_m is non-empty.

Let H be a hexagonal system, s_0 be a hexagon of H, O be the center of s_0. Draw three straight lines through O such that every line perpendicularly intersects a group of parallel edges of H. In fact, we obtain six half lines denoted as OA_1, OA_2, OB_1, OB_2, OC_1 and OC_2 (see Fig. 5). We call OA_i–OB_i–OC_i $(i = 1, 2)$ as coordinate system with respect to (briefly, w.r.t) s_0. Evidently, a coordinate system OA_i–OB_1–OC_i w.r.t. s_0 divides the plane into three areas A_iOB_i, B_iOC_i, C_iOA_i.

Let OA–OB–OC be a coordinate system w.r.t. s_0. For a point w lying in some area, say AOB, we define the coordinates of w to be the lengths of OW_A and OW_B (see Fig. 6), and denote $W(OA)$ and $W(OB)$, respectively.

A characteristic graph $T(H)$ of a hexagonal system H is defined to be the graph where vertex set is the set of the hexagons of H, and where two hexagons are joined by an edge provided they have a common edge. In fact, there is a natural way to draw the graph $T(H)$ of H. We can always let the vertices of $T(H)$ to be

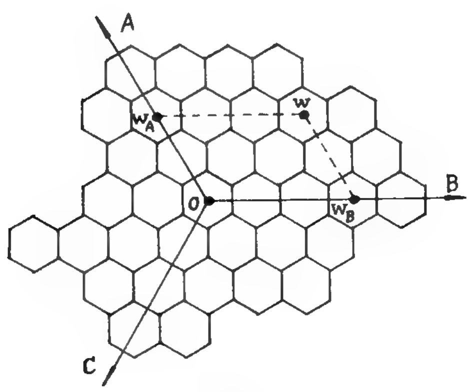
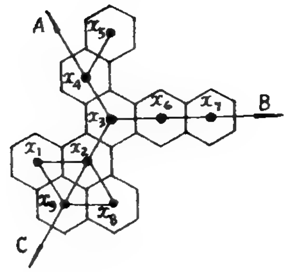


Fig. 6.



The boundary of T(H) is

$x_1x_2x_3x_4x_5x_4x_3x_6x_7x_6x_3x_2x_8x_9x_1$.

Fig. 7.

the centers of hexagons of H, two centers O_i and O_j are joined by edge provided the corresponding hexagons have a common edge (see Fig. 7).

Evidently, $T(H)$ is a planar graph. We define the perimeter of $T(H)$ to be the boundary of the exterior face, i.e. a close walk in which each cut edge of $T(H)$ is traversed twice (see Fig. 7).

Let OA–OB–OC be a coordinate system of H. If the boundary of $T(H)$ lying in some area, say AOB, is a path $W_1W_2 \ldots W_t$ after deleting the edges lying in OA and OB and the path satisfies $W_1(OA) \geq W_2(OA) \geq \cdots \geq W_t(OA)$ and $W_1(OB) \leq W_2(OB) \leq \cdots \leq W_t(OB)$; or $W_1(OA) \leq W_2(OA) \leq \cdots \leq W_t(OA)$ and $W_1(OB) \geq W_2(OB) \geq \cdots \geq W_t(OB)$, then we call the perimeter of $T(H)$ to be monotone in area AOB. If the perimeter of $T(H)$ is monotone in all three areas, then $T(H)$ is called to be monotone w.r.t. the coordinate system OA–OB–OC (see Fig. 8).

Let M be a perfect matching of H. M is called to be 3-dividable w.r.t. the coordinate system OA-OB-OC provided any edge of M does not intersect the lines OA, OB and OC, and two edges of M lie in the same area iff they are parallel (see Fig. 9).

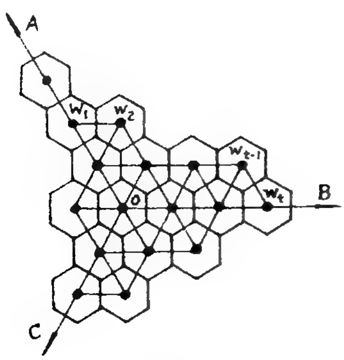


Fig. 8.

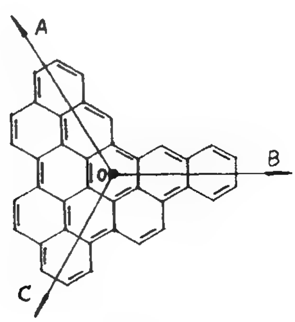


Fig. 9.

Lemma 5. *Let H be a hexagonal system, v be a vertex lying on the perimeter of H. If H or $H - v$ has a perfect matching M, then there is a hexagon which is an M-alternating cycle.*

Proof. by Lemma 2.1 in [11] or Lemma 2 in [13], the lemm is immediate. □

The following theorem will describe those hexagonal system whose Z-transformation graphs have one monovalecy vertex.

Theorem 6. *Let H be a hexagonal system. The following three statements are equivalent.*

(i) *$Z(H)$ has one monovalency vertex.*

(ii) *There exist a hexagon s_0 and a coordinate system OA–OB–OC w.r.t. s_0, and a perfect matching M of H such that M is 3-dividable w.r.t. the coordinate system OA-OB–OC.*

(iii) *There exist a hexagon s_0 and a coordinate system OA–OB–OC w.r.t. s_0 such that the perimeter of $T(H)$ is monotone w.r.t. the coordinate system OA–OB–OC.*

Proof. $(ii) \Rightarrow (i)$ is evident.

$(i) \Rightarrow (ii)$.

Let M be a monovalency vertex of $Z(H)$, s_0 be the only hexagon which is a M-alternating cycle. Let OA–OB–OC be the coordinate system w.r.t. s_0 such that each of the three edges e_1, e_2 and e_3 of $M \cap s_0$ does not intersect the lines OA, OB and OC.

Suppose that M is not 3-dividable w.r.t. OA–OB–OC. Then in at least one area, say AOB, there is an edge $e \in M$ which is not parallel to e_1 (see Fig. 10). Since M is a monovalence vertex of $Z(H)$, it is not difficult to see that H has a series of hexagons $s_1, \ldots, s_t$ such that $e^* \in s_t$ lies on the perimeter of H, and the

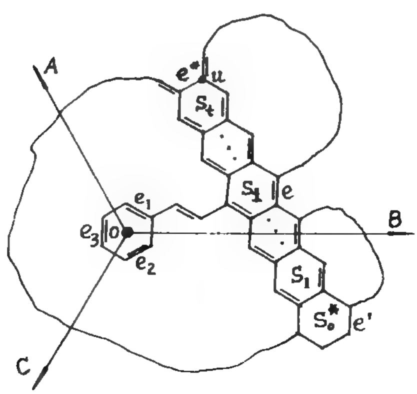
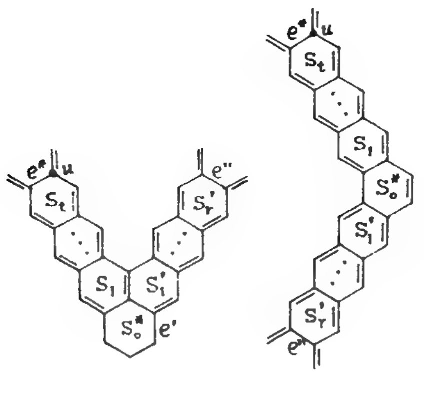


Fig. 10.



(1) (2)

Fig. 11.

vertices of s_i $(1 \leqslant i \leqslant t)$ are matched by the edges of M as depicted in Fig. 10. Let X be the set of all the hexagons of H. If the hexagon $s_0^* \notin H$, then the component H' of $H[X - \{s_1, \ldots, s_t\}]$ which contains the vertex u is a hexagonal system satisfying the conditions in Lemma 5. Hence there is another hexagon which is M-alternating cycle. This contradicts that M is *a* monovalency vertex in $Z(H)$.

If $s_0^* \in H$, according to $e' \notin M$ or $e' \in M$ (see Fig. 11), there are two possible cases, as shown in Fig. 11, where e'' lies on the perimeter of H. Let H' be the component of $H[X - \{s_1, \ldots, s_t, s_0^*, s_1', \ldots, s_r'\}]$ which contains the vetex u. By analogous arguments as above, both cases are impossible. Thus M is 3-dividable w.r.t. OA–OB–OC.

$(ii) \Rightarrow (iii)$

By contradiction. Suppose that s_0, M and OA–OB–OC mean just the same as in the statement (ii). But the perimeter of $T(H)$ is not monotone w.r.t. OA–OB-OC. Without loss of generality, we may assume in area AOB there are two vertices O_i and O_j of $T(H)$ lying on the perimeter of $T(H)$ with $O_i(OA) < O_j(OA)$ and $O_i(OB) < O_j(OB)$; moreover, edges e and e' are on the perimeter of H (see Fig. 12).

We can find a series of hexagons of H $s_j, s_1, \ldots, s_t$, $(t \geqslant 0)$, such that the edge e'' is on the perimeter of H. Let the component of $H[X - \{s_j, s_1, \ldots, s_t\}]$ containing the hexagon s_j' be H_j. Evidently, H_j satisfies the conditions in Lemma 5 hence contains a hexagon which is a M-alternating cycle. This contradicts the statement (ii).

$(iii) \Rightarrow (ii)$

By induction on the number of hexagons of H.

Suppose that there is a coordinate OA–OB–OC w.r.t. a hexagon s_0 of H such that the perimeter of $T(H)$ is monotone w.r.t. the coordinate system OA–OB–OC. If H contains only one hexagon, there is nothing to prove. We now suppose that H contains more than one hexagon. If the vertices of $T(H)$ all lie on the axes

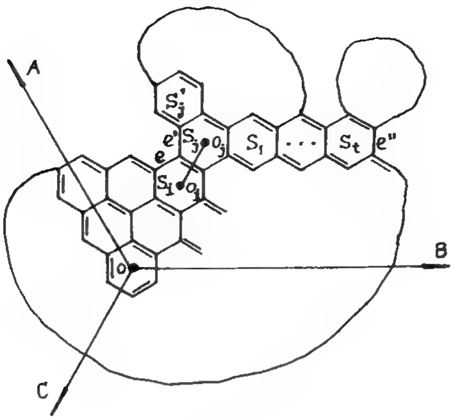


Fig. 12.

OA, OB and OC, then it is easy to check that the statement (ii) hold. We now assume that there is at least one vertex v on the perimeter of $T(H)$ not lying on the axes. Without loss of generality, we may assume v is in the area AOB (see Fig. 13), and $v'(OA)=v(OA)$, $v'(OB)<v(OB)$, $v''(OA)<v(OA)$, $v''(OB)=v(OB)$, where v' and v'' lie on the perimeter of $T(H)$ and are adjacent to v.

Let s_i be the hexagon of H whose center is v. Evidently the perimeter of $H[X-\{s_i\}]$ is still monotone w.r.t. the coordinate system OA–OB–OC. By the inductive hypothesis there exists a perfect matching M' which is 3-dividable w.r.t. OA–OB–OC. Therefore the perfect matcing $M' \cup e'$ of H is 3-dividable w.r.t. the coordinate system OA–OB–OC. This completes our proof. □

In the following we shall give a more intuitive description for a hexagonal system H whose Z-transformation graph has a monovalency vertex. Note that for any hexagonal system H there exists a smallest hexagon with its edges parallel to the edge of H and containing the interior of H in its interior. We denote it by $S(H)$.

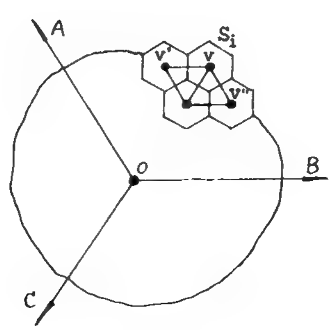


Fig. 13.

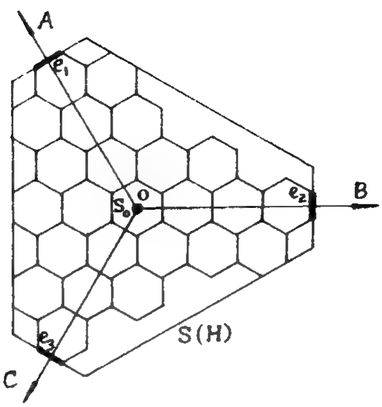


Fig. 14.

Theorem 7. *Let H be a hexagonal system. Then $Z(H)$ contains a monovalency vertex iff the following three conditions are satisfied.*

(1) *There are three pairwise disjoint edges of $S(H)$ each of which contains exactly an edge e_i of H $(i = 1, 2, 3)$.*

(2) *The perpendicular bisectors of e_1, e_2 and e_3 intersect at the center 0 of some hexagon s_0 of H.*

(3) *The perimeter of $T(H)$ is monotone w.r.t. the coordinate system OA–OB–OC shown in Fig.* 14.

Proof. By Theorem 6, it is easy to verify that if the conditions (1)–(3) are satisfied, then $Z(H)$ has a monovalency vertex.

Conversely, assume that $Z(H)$ has a monovalency vertex M. By Theorem 6, there exist a hexagon s_0 of H and a coordinate system OA–OB–OC w.r.t. s_0 such that the perimeter of $T(H)$ is monotone w.r.t. OA–OB–OC. Moreover M is 3-dividable w.r.t. OA–OB–OC. It is not difficult to see that the edge e_i which is intersected by one of the coordinate axes OA, OB and OC and is the farthest from the center of s_0 must be contained in an edge L_i of $S(H)$ and L_i contains only one such edge of M. Evidently, the axis intersecting e_i is the perpendicular and bisector of e_i.

Now the proof is completed. □

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